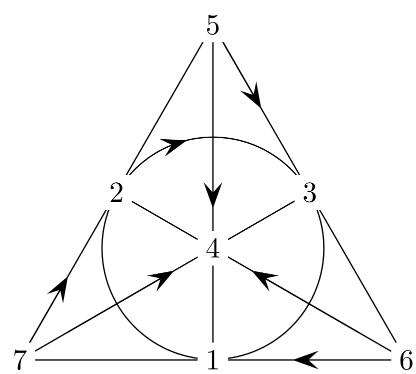
Degeneracy loci and G_2 flags
(February 8, 2009)

David E. Anderson



5
2
3
4
7
1
6

Acknowledgements

Graduate school, for me, has been an intense and sometimes difficult experience. I'd like to start by thanking all the friends in Ann Arbor who made that experience more fun, especially Elizabeth Coe, Leo Goldmakher, Hester Graves, Mark Iwen, Ryan Kinser, Mike Lieberman, and Craig Spencer. Special thanks also to my parents and sister for stomaching a couple overzealous explanations of certain agro-mathematical metaphors. Most of all, I thank Liz Vivas for her love and support through both the brightest and darkest times of the last few years.

The math department at the University of Michigan is a remarkably congenial place to study and work, and I want to thank all the people who make it so. It has been a great pleasure to learn mathematics from and with many people here and elsewhere, especially Karcsi Böröczky, Chuck Cadman, Renzo Cavalieri, Linda Chen, Brian Conrad, Igor Dolgachev, Danny Gillam, Milena Hering, Ben Howard, Robert Lazarsfeld, Leonardo Mihalcea, Ezra Miller, Yogesh More, Mircea Mustaţă, Karen Smith, Alan Stapledon, John Stembridge, Tamás Szamuely, Ágnes Szilárd, Michael Thaddeus, Kevin Tucker, Giancarlo Urzúa, Ravi Vakil, and Alex Yong. I want to thank Sam Payne and Julianna Tymoczko in particular for encouraging my interest in combinatorial algebraic geometry, and for their advice along the way.

While working on this thesis, I also benefitted from emails and discussions with Robert Bryant, Skip Garibaldi, Bill Graham, Gopal Prasad, and Richárd Rimányi.

I first learned about Schubert varieties from Sergey Fomin, and his lectures inspired me to learn about Schubert calculus. I thank him for teaching me how to think about combinatorics, as well as for his candid advice and comments on manuscripts.

Finally, I am deeply grateful to Bill Fulton for generously sharing his time and thoughts, for suggesting this project and others, for his encouragement while I pursued them, and for detailed comments on the manuscripts that resulted. He has given me invaluable advice at every stage of graduate school, and I thank him for showing me how to be a mathematician.

Contents

Acknowledgements iv

Chapter 1. Introduction 1
1.1. Overview 4

Chapter 2. Octonions and compatible forms 11
2.1. Standard facts 11
2.2. Forms 12
2.3. Octonion bundles 17
2.4. Another construction 18

Chapter 3. Isotropic flags and flag bundles 21
3.1. Topology of G_2 flags 21
3.2. Cohomology of flag bundles 24

Chapter 4. Giambelli formulas for Schubert loci 29
4.1. Divided difference operators and Chern class formulas 29
4.2. Variations 31

Chapter 5. Degeneracy of morphisms 37
5.1. Triality symmetry 38
5.2. Orbits 38
5.3. Graphs 41

Chapter 6. Characteristic two 45
6.1. Forms and octonions 45
6.2. Forms and homogeneous spaces 48

Appendix A. Lie theory 51
A.1. General facts 51
A.2. Representation theory of G_2 52
A.3. The Weyl group 55
A.4. Homogeneous spaces 56
A.5. The Borel map and divided differences 57
A.6. Equivariant vector bundles 58

Appendix B. Triality 61
B.1. D_4 flags 61
B.2. Geometric triality 63

B.3. Root systems 64
B.4. Tangent spaces 66

Appendix C. Graphs and symmetry 67
C.1. The exponential map 67
C.2. Two methods 69
C.3. General setup 70

Appendix D. Tables 75
D.1. Parametrizations of Schubert cells 75
D.2. Formulas 77
D.3. Localizations of equivariant Schubert classes 80
D.4. Multiplication table of $H_T^*(Fl_\gamma, \mathbb{Z})$ 83

Appendix E. Integral Chow rings of quadric bundles 87

Bibliography 91

CHAPTER 1

Introduction

Let V be an n-dimensional vector space. The *flag variety* $Fl(V)$ parametrizes all complete flags in V, i.e., saturated chains of subspaces $E_\bullet = (E_1 \subset E_2 \subset \cdots \subset E_n = V)$ (with $\dim E_i = i$). Fixing a flag $F_\bullet$ allows one to define *Schubert varieties* in $Fl(V)$ as the loci of flags satisfying certain incidence conditions with $F_\bullet$; there is one such Schubert variety for each permutation of $\{1, \ldots, n\}$. This generalizes naturally to the case where V is a vector bundle and $F_\bullet$ is a flag of subbundles. Here one has a *flag bundle* $\mathbf{Fl}(V)$ over the base variety, whose fibers are flag varieties, with *Schubert loci* defined similarly by incidence conditions. Formulas for the cohomology classes of these Schubert loci, as polynomials in the Chern classes of the bundles involved, include the classical Thom–Porteous–Giambelli and Kempf–Laksov formulas (see [**Fu1**]).

The above situation is "type A," in the sense that $Fl(V)$ is isomorphic to the homogeneous space SL_n/B (with B the subgroup of upper-triangular matrices). There are straightforward generalizations to the other classical types (B, C, D): here the vector bundle V is equipped with a symplectic or nondegenerate symmetric bilinear form, and the flags are required to be isotropic with respect to the given form. Schubert loci are defined as before, with one for each element of the corresponding Weyl group. The problem of finding formulas for their cohomology classes has been studied by Harris–Tu [**Ha-Tu**], Józefiak–Lascoux–Pragacz [**Jó-La-Pr**], and Fulton [**Fu2, Fu3**], among others.

One is naturally led to consider the analogous problem in the five remaining Lie types. In exceptional types, however, it is not so obvious how the Lie-theoretic geometry of G/B generalizes to the setting of vector bundles in algebraic geometry. The primary goal of this thesis is to carry this out for type G_2.

To give a better idea of the difference between classical and exceptional types, let us describe the classical problem in slightly more detail. The flag bundles are the universal cases of general *degeneracy locus* problems in algebraic geometry. Specifically, let V be a vector bundle of rank n on a variety X, and let $\varphi : V \otimes V \to k$ be a symplectic or nondegenerate symmetric bilinear form (or the zero form). If $E_\bullet$ and $F_\bullet$ are general flags of isotropic subbundles of V, the problem is to find formulas in H^*X for the *degeneracy locus*

$$D_w = \{x \in X \mid \dim(F_p(x) \cap E_q(x)) \geq r_{w\,w_0}(q,p)\},$$

in terms of the Chern classes of the line bundles E_q/E_{q-1} and F_p/F_{p-1}, for all p and q. (Here w is an element of the Weyl group, considered as a permutation via an embedding in the symmetric group S_n; w_0 is the longest element, corresponding to the permutation $n\ n-1\ \cdots\ 1$; and $r_w(q,p) = \#\{i \leq q \,|\, w(i) \leq p\}$ is a nonnegative integer depending on w, p, and q.) Such formulas have a wide range of applications: for example, they appear in the theory of special divisors and variation of Hodge structure on curves in algebraic geometry [**Ha-Tu, Pa-Pr**], and they are used to study singularities of smooth maps in differential geometry (work of Fehér and Rimányi). They are also of interest in combinatorics (Lascoux–Schützenberger, Fomin–Kirillov, Pragacz, Kresch–Tamvakis). See [**Fu-Pr**] for a more detailed account of the history.

In this thesis, we pose and solve the corresponding problem in type G_2:

> *Let $V \to X$ be a vector bundle of rank 7, equipped with a nondegenerate alternating trilinear form $\gamma : \bigwedge^3 V \to L$, for a line bundle L. Let $E_\bullet$ and $F_\bullet$ be general flags of γ-isotropic subbundles of V, and let*
>
> $$D_w = \{x \in X \mid \dim(F_p(x) \cap E_q(x)) \geq r_{w\,w_0}(q,p)\},$$
>
> *where w is an element of $W(G_2)$ (the dihedral group with 12 elements), and $r_w(q,p)$ is a certain nonnegative integer. Find a formula for $[D_w]$ in H^*X, in terms of the Chern classes of the bundles involved.*

(The meaning of "nondegenerate" and "γ-isotropic" will be explained below (§§1.1.1–1.1.2), as will the precise definition of D_w (§1.1.5).) Proofs and variations of the formulas are given in Chapter 4; the formulas themselves are listed in Appendix D.2.

Formulas for degeneracy loci are closely related to Giambelli formulas for equivariant classes of Schubert varieties in the equivariant cohomology of the corresponding flag variety. We will usually use the language of degeneracy loci, but we discuss the connection with equivariant cohomology in §1.1.6. In brief, the two perspectives are equivalent when $\det V$ and L are trivial line bundles.

Another notion of degeneracy loci is often useful. Let $\varphi : E \to F$ be a morphism of vector bundles on X, subject to some symmetry hypothesis; for example, if $F = E^*$, one may require that $\varphi^* : E^{**} = E \to E^*$ be equal to φ. In this setup, there is a *degeneracy locus*

$$D_r = D_r(\varphi) = \{x \in X \mid \mathrm{rk}(\varphi(x) : E(x) \to F(x)) \leq r\},$$

and one can ask for formulas for such loci as well. (The expected codimension of D_r depends on the type of symmetry one imposes on φ.) In classical types, this problem is equivalent to the "incidence" version discussed above, by replacing φ with its graph in $V = E \oplus F$. Indeed, many of the works cited above deal with morphisms rather than subbundles.

Once again, it makes sense to consider this problem in exceptional types, but the appropriate notion of symmetry is somewhat more subtle than in classical types. In Appendix C, we introduce a general notion of symmetry corresponding to a homogeneous space G/P, and discuss its relation to the graph of a morphism. Chapter 5 gives formulas for degeneracy loci of *triality-symmetric* morphisms, which is the G_2 case; see §1.1.7 for the definition. Here we will deal with equivariant cohomology more directly: in the spirit of [**Fe-Né-Ri**], these formulas come from equivariant classes of orbit closures in a vector space.

When the base X is a point, so V is a vector space and the flag bundle is just the flag variety G/B, most of the results have been known for some time; essentially everything can be done using the general tools of Lie theory. For example, a presentation of $H^*(G/B, \mathbb{Z})$ was given by Bott and Samelson [**Bo-Sa**], and (different) formulas for Schubert classes in $H^*(G/B, \mathbb{Q})$ appear in [**BGG**]. Since this thesis also aims to present a concrete, unified perspective on the G_2 flag variety, accessible to general algebraic geometers, we wish to emphasize geometry over Lie theory: we are describing a geometric situation from which type-G_2 groups arise naturally. Reflecting this perspective, we use Lie-theoretic arguments sparingly, avoiding them altogether in the first four chapters. Appendix A collects the basic representation-theoretic facts we use, and relates our geometric constructions with the general Lie-theoretic ones. In Appendix B, we give a brief exposition of triality and its relation to the G_2 flag variety.

Various constructions of exceptional-type flag varieties have been given using techniques from algebra and representation theory; those appearing in [**La-Ma**], [**Il-Ma**], and [**Ga2**] have a similar flavor to the one presented here. A key feature of our description is that the data parametrized by the G_2 flag variety naturally determine a *complete* flag in a 7-dimensional vector space, much as isotropic flags in classical types determine complete flags by taking orthogonal complements. The fundamental facts that make this work are Proposition 1.1.2 and its cousins, Corollary 2.2.10 and Propositions 2.3.2 and 2.3.3.

Notation and conventions. Unless otherwise indicated, the base field k will have characteristic not 2 and be algebraically closed (although a quadratic extension of the prime field usually suffices). Angle brackets denote the span of enclosed vectors: $\langle x, y, z\rangle := \mathrm{span}\{x, y, z\}$. If $X \to Y$ is a morphism and V is a vector bundle on Y, we will often write V for the vector bundle pulled back to X; if x is a point of X, $V(x)$ denotes the fiber over x. If V is a vector space and E is a subspace, $[E]$ denotes the corresponding point in an appropriate Grassmannian. For a group G, if X is a right G-space and Y is a left G-space, we write $X \times^G Y$ for the "balanced quotient," given by the equivalence relation $(x, g \cdot y) \sim (x \cdot g, y)$.

We generally use the notation and language of (singular) cohomology, but this should be read as Chow cohomology for ground fields other than $\mathbb{C}$.

(Since the varieties whose cohomology we compute are rational homogeneous spaces or fibered in homogeneous spaces, the distinction is not significant.)

1.1. Overview

We begin with an overview of our description of the G_2 flag variety and statements of the main results. Proofs and details are given in later sections.

1.1.1. Compatible forms. Let V be a k-vector space. Let β be a nondegenerate symmetric bilinear form on V, and let γ be an alternating trilinear form, i.e., $\gamma : \bigwedge^3 V \to k$. Write $v \mapsto v^\dagger$ for the isomorphism $V \to V^*$ defined by β, and $\varphi \mapsto \varphi^\dagger$ for the inverse map $V^* \to V$. (Explcitly, these are defined by $v^\dagger(u) = \beta(v,u)$ and $\varphi(u) = (\varphi^\dagger, u)$ for any $u \in V$.) Our constructions are based on the following definitions:

Definition 1.1.1. Call the forms γ and β **compatible** if

$$2\,\gamma(u,v,\gamma(u,v,\cdot)^\dagger) = \beta(u,u)\beta(v,v) - \beta(u,v)^2 \tag{1.1.1}$$

for all $u,v \in V$. An alternating trilinear form $\gamma : \bigwedge^3 V \to k$ is **nondegenerate** if there exists a compatible nondegenerate symmetric bilinear form on V.

The meaning of the strange-looking relation (1.1.1) will be explained in §2; see Proposition 2.2.1. (The factor of 2 is due to our convention that a quadratic norm and corresponding bilinear form are related by $\beta(u,u) = 2\,N(u)$.) A pair of compatible forms is equivalent to a composition algebra structure on $k\oplus V$ (see §2). Since a composition algebra must have dimension 1, 2, 4, or 8 over k (by Hurwitz's theorem), it follows that nondegenerate trilinear forms exist only when V has dimension 1, 3, or 7. In each case, there is an open dense $GL(V)$-orbit in $\bigwedge^3 V^*$ consisting of nondegenerate forms. When $\dim V = 1$, the only alternating trilinear form is zero, and any nonzero bilinear form is compatible with it. When $\dim V = 3$, an alternating trilinear form is a scalar multiple of the determinant, and given a nondegenerate bilinear form, it is easy to show that there is a unique compatible trilinear form up to sign.

When $\dim V = 7$, it is less obvious that $\bigwedge^3 V^*$ has an open $GL(V)$-orbit, especially if $\mathrm{char}(k) = 3$, but it is still true (Proposition A.2.2). The choice of γ determines β uniquely up to scalar — in fact, up to a cube root of unity (see Proposition A.2.6).

Associated to any alternating trilinear form γ on a seven-dimensional vector space V, there is a canonical map $B_\gamma : \mathrm{Sym}^2 V \to \bigwedge^7 V^*$, determining (up to scalar) a bilinear form β_γ. We will give the formula for $\mathrm{char}(k) \neq 3$ here. Following Bryant [**Br**], we define B_γ by

$$B_\gamma(u,v) = -\frac{1}{3}\gamma(u,\cdot,\cdot) \wedge \gamma(v,\cdot,\cdot) \wedge \gamma, \tag{1.1.2}$$

where $\gamma(u,\cdot,\cdot) : \bigwedge^2 V \to k$ is obtained by contracting γ with u. Choosing an isomorphism $\bigwedge^7 V^* \cong k$ yields a symmetric bilinear form β_γ. If β_γ is nondegenerate, then a scalar multiple of it is compatible with the trilinear form γ; thus γ is nondegenerate if and only if β_γ is nondegenerate. The form β_γ is defined in characteristic 3, as well, and the statement still holds (see Lemma 2.2.7 and its proof).

1.1.2. Isotropic spaces. For the rest of this section, assume $\dim V = 7$. Given a nondegenerate trilinear form γ on V, say a subspace F of dimension at least 2 is **γ-isotropic** if $\gamma(u,v,\cdot) \equiv 0$ for all $u,v \in F$. (That is, the map $F \otimes F \to V^*$ induced by γ is zero.) Say a vector or a 1-dimensional subspace is γ-isotropic if it is contained in a 2-dimensional γ-isotropic space. If β is a compatible bilinear form, every γ-isotropic subspace is also β-isotropic (Lemma 2.2.3); as usual, this means β restricts to zero on F. Since β is nondegenerate, a maximal β-isotropic subspace has dimension 3.

Proposition 1.1.2. *For any (nonzero) isotropic vector $u \in V$, the space*

$$E_u = \{v \mid \langle u, v\rangle \text{ is } \gamma\text{-isotropic}\}$$

is three-dimensional and β-isotropic. Moreover, every two-dimensional γ-isotropic subspace of E_u contains u.

The proof is given at the end of §2.2. The proposition implies that a maximal γ-isotropic subspace has dimension 2, and motivates the central definition:

Definition 1.1.3. A **γ-isotropic flag** (or G_2 **flag**) in V is a chain

$$F_1 \subset F_2 \subset V$$

of γ-isotropic subspaces, of dimensions 1 and 2. The variety parametrizing γ-isotropic flags is called the **γ-isotropic flag variety** (or G_2 **flag variety**), and denoted $Fl_\gamma(V)$.

The γ-isotropic flag variety is a smooth, six-dimensional projective variety (Proposition 3.1.1). See §A.4 for its description as a homogeneous space.

Proposition 1.1.2 shows that a γ-isotropic flag has a unique extension to a complete flag in V: set $F_3 = E_u$ for u spanning F_1, and let F_{7-i} be the orthogonal space $F_i^\perp$, with respect to a compatible form β. (Since a compatible form is unique up to scalar, this is independent of the choice of β.) This defines a closed immersion $Fl_\gamma(V) \hookrightarrow Fl_\beta(V) \subset Fl(V)$, where $Fl_\beta(V)$ and $Fl(V)$ are the (classical) type B and type A flag varieties, respectively.

From the definition, there is a tautological sequence of vector bundles on $Fl_\gamma(V)$,

$$S_1 \subset S_2 \subset V,$$

and this extends to a **complete γ-isotropic flag** of bundles

$$S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5 \subset S_6 \subset V$$

by the proposition. Similarly, there are universal quotient bundles $Q_i = V/S_{7-i}$.

1.1.3. Bundles. Now let $V \to X$ be a vector bundle of rank 7, and let L be a line bundle on X. An alternating trilinear form $\gamma : \bigwedge^3 V \to L$ is **nondegenerate** if it is locally nondegenerate on fibers. Equivalently, we may define the Bryant form $B_\gamma : \mathrm{Sym}^2 V \to \det V^* \otimes L^{\otimes 3}$ by Equation (1.1.2), and γ is nondegenerate if and only if B_γ is (so B_γ defines an isomorphism $V \cong V^* \otimes \det V^* \otimes L^{\otimes 3}$).

A subbundle F of V is **γ-isotropic** if each fiber $F(x)$ is γ-isotropic in $V(x)$; for F of rank 2, this is equivalent to requiring that the induced map $F \otimes F \to V^* \otimes L$ be zero. If $F_1 \subset V$ is γ-isotropic, the bundle $E_{F_1} = \ker(V \to F_1^* \otimes V^* \otimes L)$ has rank 3 and is isotropic for B_γ. (If u is a vector in a fiber $F_1(x)$, then $E_{F_1}(x) = E_u$, in the notation of §1.1.2.)

Given a nondegenerate form γ on V, there is a **γ-isotropic flag bundle** $\mathbf{Fl}_\gamma(V) \to X$, with fibers $Fl_\gamma(V(x))$. This comes with universal γ-isotropic subbundles S_i and quotient bundles Q_i, as before.

1.1.4. Chern class formulas. In the setup of §1.1.3, one has Schubert loci $\mathbf{\Omega}_w \subseteq \mathbf{Fl}_\gamma(V)$ indexed by the Weyl group. There is an embedding of $W = W(G_2)$ in the symmetric group S_7 such that the permutation corresponding to $w \in W$ is determined by its first two values. We identify w with this pair of integers, so $w = w_1\, w_2$; see §A.3 for more on the Weyl group. As in classical types, we set

$$r_w(q,p) = \#\{i \leq q \,|\, w(i) \leq p\}. \tag{1.1.3}$$

Given a fixed γ-isotropic flag $F_\bullet$ on X, the Schubert loci are defined by

$$\mathbf{\Omega}_w = \{x \in \mathbf{Fl}_\gamma(V) \,|\, \mathrm{rk}(F_p \to Q_q) \leq r_w(q,p) \text{ for } 1 \leq p \leq 7,\ 1 \leq q \leq 2\}.$$

These are locally trivial fiber bundles, whose fibers are Schubert varieties in $Fl_\gamma(V(x))$.

The G_2 **divided difference operators** ∂_s and ∂_t act on $\Lambda[x_1, x_2]$, for any ring Λ, by

$$\partial_s(f) = \frac{f(x_1,x_2) - f(x_2,x_1)}{x_1 - x_2}; \tag{1.1.4}$$

$$\partial_t(f) = \frac{f(x_1,x_2) - f(x_1,x_1-x_2)}{-x_1 + 2x_2}. \tag{1.1.5}$$

If $w \in W$ has reduced word $w = s_1 \cdot s_2 \cdots s_\ell$ (where s_i is the simple reflection s or t), then define ∂_w to be the composition $\partial_{s_1} \circ \cdots \circ \partial_{s_\ell}$. This is independent of the choice of word; see §A.5. (As mentioned in §A.3, each $w \in W(G_2)$ has a unique reduced word, with the exception of w_0, so independence of choice is actually lack of choice in this case.) These formulas also define operators on $H^*\mathbf{Fl}_\gamma(V)$. (See §4.1.)

Let V be a vector bundle of rank 7 on X equipped with a nondegenerate form $\gamma : \bigwedge^3 V \to k_X$, and assume $\det V$ is trivial. Let $F_1 \subset F_2 \subset \cdots \subset V$ be a complete γ-isotropic flag in V. Set $y_1 = c_1(F_1)$, $y_2 = c_1(F_2/F_1)$. Let $\mathbf{Fl}_\gamma(V) \to X$ be the flag bundle, and set $x_1 = -c_1(S_1)$ and $x_2 = -c_1(S_2/S_1)$, where $S_1 \subset S_2 \subset V$ are the tautological bundles.

THEOREM 1.1.4. *We have*

$$[\mathbf{\Omega}_w] = P_w(x;y),$$

where $P_w = \partial_{w_0\, w^{-1}} P_{w_0}$, *and*

$$P_{w_0}(x;y) = \frac{1}{2}(x_1^3 - 2\,x_1^2\,y_1 + x_1\,y_1^2 - x_1\,y_2^2 + x_1\,y_1\,y_2 - y_1^2\,y_2 + y_1\,y_2^2) \\ \times (x_1^2 + x_1\,y_1 + y_1\,y_2 - y_2^2)(x_2 - x_1 - y_2).$$

in $H^*(\mathbf{Fl}_\gamma(V),\mathbb{Z})$. *(Here* w_0 *is the longest element of the Weyl group.)*

The proof is given in §4.1, along with a discussion of alternative formulas, including ones where γ takes values in $M^{\otimes 3}$ for an arbitrary line bundle M.

1.1.5. Degeneracy loci. Returning to the problem posed in the introduction, let V be a rank 7 vector bundle on a variety X, with nondegenerate form γ and two (complete) γ-isotropic flags of subbundles $F_\bullet$ and $E_\bullet$. The first flag, $F_\bullet$, allows us to define Schubert loci in the flag bundle $\mathbf{Fl}_\gamma(V)$ as in §1.1.4. The second flag, $E_\bullet$, determines a section s of $\mathbf{Fl}_\gamma(V) \to X$, and we define degeneracy loci as scheme-theoretic inverse images under s:

$$D_w = s^{-1}\mathbf{\Omega}_w \subset X.$$

When X is Cohen-Macaulay and D_w has expected codimension (equal to the length of w; see §A.3), we have

$$[D_w] = s^*[\mathbf{\Omega}_w] = P_w(x;y) \tag{1.1.6}$$

in H^*X, where $x_i = -c_1(E_i/E_{i-1})$ and $y_i = c_1(F_i/F_{i-1})$. More generally, this polynomial defines a class supported on D_w, even without assumptions on the singularities of X or the genericity of the flags $F_\bullet$ and $E_\bullet$; see [**Fu1**] or [**Fu-Pr**, App. A] for the intersection-theoretic details.

1.1.6. Equivariant cohomology. Now return to the case where V is a 7-dimensional vector space. One can choose a basis $f_1,\ldots,f_7$ such that $F_i = \langle f_1,\ldots,f_i\rangle$ forms a complete γ-isotropic flag in V, and let $T = (k^*)^2$ act on $V \cong k^7$ by

$$(z_1,z_2) \mapsto \mathrm{diag}(z_1,z_2,z_1z_2^{-1},1,z_1^{-1}z_2,z_2^{-1},z_1^{-1}).$$

Write t_1 and t_2 for the corresponding weights. Then T preserves γ and acts on $Fl_\gamma(V)$. The total equivariant Chern class of V is $c^T(V) = (1-t_1^2)(1-t_2^2)(1-(t_1-t_2)^2)$, so we have

$$H_T^*(Fl_\gamma(V),\mathbb{Z}[\tfrac{1}{2}]) = \mathbb{Z}[\tfrac{1}{2}][x_1,x_2,t_1,t_2]/(r_2,r_4,r_6),$$

with the relations $r_{2i} = e_i(x_1^2,\, x_2^2,\, (x_1-x_2)^2) - e_i(t_1^2,\, t_2^2,\, (t_1-t_2)^2)$. A presentation with $\mathbb{Z}$ coefficients can be deduced from Theorem 3.2.4; see Remark 3.2.5.

Theorem 1.1.4 yields an equivariant Giambelli formula:

$$[\Omega_w]^T = P_w(x;t) \quad \text{in } H_T^*Fl_\gamma.$$

In fact, this formula holds with integer coefficients: the Schubert classes form a basis for $H_T^*(Fl_\gamma, \mathbb{Z})$ over $\mathbb{Z}[t_1, t_2]$, so in particular there is no torsion, and $H_T^*(Fl_\gamma, \mathbb{Z})$ includes in $H_T^*(Fl_\gamma, \mathbb{Z}[\frac{1}{2}])$.

Given an equivariant Giambelli formula, one can easily find the localization of a Schubert class $[\Omega_w]^T$ at a fixed point $e(v)$ and compute the multiplication table of $H_T^*(Fl_\gamma)$ with respect to the Schubert basis. These computations are given in §D.3 and §D.4.

The equivariant geometry of Fl_γ is closely related to the degeneracy loci problem; we briefly describe the connection. In the setup of §1.1.5, assume V has trivial determinant and γ has values in the trivial bundle, so the structure group is $G = G_2$. The data of two γ-isotropic flags in V gives a map to the classifying space $BB \times_{BG} BB$, where $B \subset G$ is a Borel subgroup, and there are universal degeneracy loci $\mathbf{\Omega}_w$ in this space. On the other hand, there is an isomorphism $BB \times_{BG} BB \cong EB \times^B (G/B)$, carrying $\mathbf{\Omega}_w$ to $EB \times^B \Omega_w$. Since $H_T^*(Fl_\gamma) = H^*(EB \times^B (G/B))$, and $[\Omega_w]^T = [EB \times^B \Omega_w]$, a Giambelli formula for $[\Omega_w]^T$ is equivalent to a degeneracy locus formula for this situation. One may then use equivariant localization to verify a given formula; this is essentially the approach taken in [**Gr2**].

1.1.7. Triality-symmetric morphisms. A nondegenerate skew-symmetric bilinear form on a vector space V of dimension $2n$ gives rise to a duality involution of the (type A) Grassmannian $Gr(n, V)$ whose fixed locus is the Lagrangian Grassmannian $LG(n, V) \subseteq Gr(n, V)$. If $E \subset V$ is an isotropic n-dimensional subspace, corresponding to a point $[E] \in LG(n, V)$, then $V = E \oplus E^*$ and the tangent spaces $T_{[E]}LG(n, V) \subseteq T_{[E]}Gr(n, V)$ may be identified with $\mathrm{Sym}^2 E^* \subseteq \mathrm{Hom}(E, V/E) = \mathrm{Hom}(E, E^*)$.

Similarly, a nondegenerate alternating trilinear form on a seven dimensional vector space V (or an octonion algebra structure on $C = k \oplus V$) gives rise to a *triality automorphism* of the type D_4 Grassmannian $OG(2, C)$, whose fixed locus is the G_2 Grassmannian $\mathcal{G}$ of γ-isotropic 2-planes in V. In this case, given a two-dimensional γ-isotropic subspace $E \subset V \subset C$, the form identifies $C = E \oplus \mathrm{End}(E) \oplus E^*$, and the tangent spaces

$$\begin{aligned} T_{[E]}\mathcal{G} &\subseteq T_{[E]}OG(2, C) \\ &\subseteq T_{[E]}Gr(2, C) \end{aligned}$$

are identified with

$$\begin{aligned} (\mathrm{Sym}^3 E^* \otimes \textstyle\bigwedge^2 E) \oplus \bigwedge^2 E^* &\subseteq (E^* \otimes E^* \otimes E^* \otimes \textstyle\bigwedge^2 E) \oplus \bigwedge^2 E^* \\ &\subseteq \mathrm{Hom}(E, \mathrm{End}(E) \oplus E^*). \end{aligned}$$

It is therefore natural to call linear maps $E \to \mathrm{End}(E) \oplus E^*$ lying in the subspace $(\mathrm{Sym}^3 E^* \otimes \bigwedge^2 E) \oplus \bigwedge^2 E^*$ *triality-symmetric* maps. (More details on triality are reviewed in Appendix B.)

The above discussion globalizes naturally to vector bundles. Let E be a rank 2 vector bundle on a variety X. Consider a morphism $\varphi : E \to$

$End(E) \oplus E^*$, corresponding to a section of $(E^* \otimes End(E)) \oplus (E^* \otimes E^*)$. Since E has rank 2, there is a canonical isomorphism $E \cong E^* \otimes \bigwedge^2 E$. Thus we can identify

$$E^* \otimes End(E) = E^* \otimes E^* \otimes E = E^* \otimes E^* \otimes E^* \otimes \bigwedge^2 E.$$

Write $\varphi = \varphi_1 \oplus \varphi_2$, with φ_1 a section of $(E^*)^{\otimes 3} \otimes \bigwedge^2 E$ and φ_2 a section of $E^* \otimes E^*$.

DEFINITION 1.1.5. A morphism $\varphi : E \to End(E) \oplus E^*$ is **triality-symmetric** if the corresponding section lies in

$$(\mathrm{Sym}^3 E^* \otimes \bigwedge^2 E) \oplus \bigwedge^2 E^*.$$

That is, $\varphi = \varphi_1 \oplus \varphi_2$, with φ_1 defining a symmetric trilinear form $\mathrm{Sym}^3 E \to \bigwedge^2 E$ and φ_2 defining an alternating bilinear form $\bigwedge^2 E \to \mathcal{O}_X$.

Write $D_r(\varphi) \subseteq X$ for the locus of points where φ has rank at most r. For a triality-symmetric morphism φ, the **expected codimension** of D_r is 5, 3, or 0 if $r = 0$, $r = 1$, or $r = 2$, respectively.

THEOREM 1.1.6. *Let c_1, c_2 be the Chern classes of E^*, and let x_1, x_2 be Chern roots. Let $\varphi : E \to End(E) \oplus E^*$ be a triality-symmetric morphism. If $D_r(\varphi)$ has expected codimension and X is Cohen-Macaulay, then we have $[D_r(\varphi)] = P_r(c_1, c_2)$ in H^*X, where*

$$\begin{aligned} P_2 &= 1, \\ P_1 &= 3\,c_2\,c_1 = 3x_1x_2(x_1 + x_2), \\ P_0 &= c_2\,c_1\,(9\,c_2 - 2\,c_1^2) = x_1x_2(x_1 + x_2)(2x_1 - x_2)(-x_1 + 2x_2). \end{aligned}$$

Two proofs are given in Chapter 5, along with formulas for other loci.

1.1.8. Problems. We conclude this overview with a brief outline of two projects naturally suggested by the present work.

1.1.8.1. *Other types.* It is reasonable to hope for a similar degeneracy locus story in some of the remaining exceptional types. Groups of type F_4 and E_6 are closely related to *Albert algebras*, and bundle versions of these algebras have been defined and studied over some one-dimensional bases [**Pu**]. Concrete realizations of the flag varieties have been given for types F_4 [**La-Ma**], E_6 [**Il-Ma**], and E_7 [**Ga2**]. Part of the challenge is to produce a complete flag from one of these realizations, and this seems to become more difficult as the dimension of the minimal irreducible representation increases with respect to the rank.

1.1.8.2. *Orbit closures in Lie algebras.* As described in §C.2.2, equivariant classes of orbit closures in $\mathfrak{g}/\mathfrak{p}$ often coincide with classes of degeneracy loci. This motivates the following problem:

> *Compute the equivariant classes of P- or B-orbit closures in $\mathfrak{g}/\mathfrak{p}$.*

Solutions to this problem account for many of the known Giambelli formulas. For example, let $G = GL_{2n}$, $P = P_n$, so $G/P = Gr(n, 2n)$ and $\mathfrak{g}/\mathfrak{p} \cong \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$. The orbits of $B \subset G$ acting on $\mathfrak{g}/\mathfrak{p}$ coincide with those of $B_n \times B_n = B \cap L$ acting on $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$, where $L = GL_n \times GL_n$ is a Levi subgroup of P, and $B_n \subset GL_n$. The latter are precisely the *matrix Schubert varieties* [**Fu1**], and their B-equivariant classes are the double Schubert polynomials of Lascoux–Schützenberger; this was proved by Fehér–Rimányi [**Fe-Ri1**] and Knutson–Miller [**Kn-Mi**].

A related problem is to classify situations where there are finitely many orbits. In the case of P acting on $\mathfrak{g}/\mathfrak{p}$, this has been done [**Bü-He, Hi-Rö, Jü-Rö**].

When does B have finitely many orbits on $\mathfrak{g}/\mathfrak{p}$? When does L have finitely many orbits, for $L \subseteq P$ a Levi subgroup?

CHAPTER 2

Octonions and compatible forms

Any description of G_2 geometry is bound to be related to *octonion algebras*, since the simple group of type G_2 may be realized as the automorphism group of an octonion algebra; see Proposition 2.1.2 below. For an entertaining and wide-ranging tour of the octonions (also known as the *Cayley numbers* or *octaves*), see [**Ba**].

The basic linear-algebraic data can be defined as in §1, without reference to octonions, but the octonionic description is equivalent and sometimes more concrete. In this chapter, we collect the basic facts about octonions that we will use, and establish their relationship with the notion of compatible forms introduced in §1.1.1. Most of the statements hold over an arbitrary field, but we will continue to assume k is algebraically closed of characteristic not 2.

While studying holonomy groups of Riemannian manifolds, Bryant proved several related facts about octonions and representations of (real forms of) G_2. In particular, he gives a way of producing a compatible bilinear form associated to a given trilinear form; we will use a version of this construction for forms on vector bundles. See [**Br**] or [**Ha**] for a discussion of the role of G_2 in differential geometry.

As far as I am aware, the results in §§2.2–2.4 have not appeared in the literature in this form, although related ideas about trilinear forms on a 7-dimensional vector space can be found in [**Br**, §2], and a construction similar to that of §2.4 is mentioned in [**Mu**].

2.1. Standard facts

Here we list some well-known facts about composition algebras, referring to [**Sp-Ve**, §1] for proofs of any non-obvious assertions.

Definition 2.1.1. A **composition algebra** is a k-vector space C with a nondegenerate quadratic norm $N : C \to k$ and an algebra structure $m : C \otimes C \to C$, with identity e, such that $N(uv) = N(u)N(v)$.

Denote by β' the symmetric bilinear form associated to N, defined by

$$\beta'(u, v) = N(u + v) - N(u) - N(v).$$

(Notice that $\beta'(u, u) = 2N(u)$; it is partly for this reason that we assume $\mathrm{char}(k) \neq 2$.) Since $N(u) = N(eu) = N(e)N(u)$ for all $u \in C$, it follows that $N(e) = 1$ and $\beta'(e, e) = 2$.

The possible dimensions for C are 1, 2, 4, and 8. A composition algebra of dimension 4 is called a **quaternion algebra**, and one of dimension 8 is an **octonion algebra**; octonion algebras are neither associative nor commutative. If there is a nonzero vector $u \in C$ with $N(u) = 0$, then C is **split**. (Otherwise C is a normed division algebra.) Any two split composition algebras of the same dimension are isomorphic. Over an algebraically closed field, C is always split, so in this case there is only one composition algebra in each possible dimension, up to isomorphism.

Define **conjugation** on C by $\overline{u} = \beta'(u, e)e - u$. Every element $u \in C$ satisfies a quadratic **minimal equation**

$$u^2 - \beta'(u, e)u + N(u)e = 0, \tag{2.1.1}$$

so

$$u\overline{u} = \overline{u}u = N(u)e. \tag{2.1.2}$$

Write $V = e^{\perp} \subset C$ for the **imaginary subspace**. For $u \in V$, $\overline{u} = -u$, so $u^2 = -N(u)e$, that is, $N(u) = -\frac{1}{2}\beta'(u^2, e)$. For $u, v \in V$, we have

$$\begin{aligned} \beta'(u, v)e &= N(u + v)e - N(u)e - N(v)e \\ &= -uv - vu. \end{aligned} \tag{2.1.3}$$

Although C may not be associative, we always have $\overline{u}(uv) = (\overline{u}u)v = N(u)v$ and $(uv)\overline{v} = u(v\overline{v}) = N(v)u$ for any $u, v \in C$. Also, for $u, v, w \in C$ we have

$$\beta'(uv, w) = \beta'(v, \overline{u}w) = \beta'(u, w\overline{v}). \tag{2.1.4}$$

A nonzero element $u \in C$ is a **zerodivisor** if there is a nonzero v such that $uv = 0$. We have $0 = \overline{u}(uv) = (\overline{u}u)v = N(u)v$, so

$$u \text{ is a zerodivisor iff } N(u) = 0. \tag{2.1.5}$$

The relevance to G_2 geometry comes from the following:

Proposition 2.1.2 ([**Sp-Ve**, §2]). *Let C be an octonion algebra over any field k. Then the group $G = \mathrm{Aut}(C)$ of algebra automorphisms of C is a simple group of type G_2, defined over k. In fact, $G \subset SO(V, \beta) \subset SO(C, \beta')$, where $V = e^{\perp}$. If $\mathrm{char}(k) \neq 2$, G acts irreducibly on V.*

2.2. Forms

The algebra structure on C corresponds to a trilinear form

$$\gamma' : C \otimes C \otimes C \to k,$$

using β' to identify C with C^*. Specifically, we have

$$\gamma'(u, v, w) = \beta'(uv, w). \tag{2.2.1}$$

Restricting γ' to V, we get an alternating form which we will denote by γ. (This follows from (2.1.4) and the fact that $\overline{u} = -u$ for $u \in V$.) Also, β' restricts to a nondegenerate form β on V, defining a canonical isomorphism $V \to V^*$.

The multiplication map $m : C\otimes C \to C$, with $C = k\oplus V$, is characterized by

$$m(u,v) = -\frac{1}{2}\beta(u,v)e + \gamma(u,v,\cdot)^\dagger \qquad \text{for } u,v\in V; \tag{2.2.2}$$

$$m(u,e) = m(e,u) = u \qquad \text{for } u\in V; \tag{2.2.3}$$

$$m(e,e) = e. \tag{2.2.4}$$

Conversely, given a trilinear form $\gamma \in \bigwedge^3 V^*$ and a nondegenerate bilinear form $\beta \in \operatorname{Sym}^2 V^*$, extend β orthogonally to $C = k\oplus V$ and define a multiplication m according to formulas (2.2.2)–(2.2.4) above.

PROPOSITION 2.2.1. *This multiplication makes C into a composition algebra with norm $N(u) = \frac{1}{2}\beta'(u,u)$ if and only if γ and β are compatible, in the sense of Definition 1.1.1.*

PROOF. This is a simple computation: For $u,v\in V$, we have

$$\begin{aligned} N(uv) &= \frac{1}{2}\beta'(uv,uv) \\ &= \frac{1}{2}\beta'(-\frac{1}{2}\beta(u,v)e, -\frac{1}{2}\beta(u,v)e) + \frac{1}{2}\beta'(\gamma(u,v,\cdot)^\dagger, \gamma(u,v,\cdot)^\dagger) \\ &= \frac{1}{4}\beta(u,v)\beta(u,v) + \frac{1}{2}\gamma(u,v,\gamma(u,v,\cdot)^\dagger), \end{aligned}$$

and

$$N(u)N(v) = \frac{1}{4}\beta(u,u)\beta(v,v).$$ □

REMARK 2.2.2. Similar characterizations of octonionic multiplication have been given, usually in terms of a *cross product* on V. See [**Br**, §2] or [**Ha**, §6].

LEMMA 2.2.3. *Suppose γ and β are compatible forms on V, defining a composition algebra structure on $C = k\oplus V$. Then $L\subset V$ is γ-isotropic iff $uv = 0$ in C for all $u,v\in L$. In particular, any γ-isotropic subspace is also β-isotropic.*

PROOF. Let γ' and β' be the forms corresponding to the algebra structure. One implication is trivial: If $uv = 0$ for all $u,v\in L$, then $\beta'(uv,\cdot) = \gamma'(u,v,\cdot) \equiv 0$ on C, so $\gamma(u,v,\cdot)\equiv 0$ on V and L is γ-isotropic.

Conversely, suppose L is γ-isotropic. First we show L is β-isotropic. Given any $u\in L$, choose a nonzero $v\in u^\perp\cap L$. Since L is γ-isotropic, $\gamma(u,v,\cdot)^\dagger = 0$, so u and v are zerodivisors:

$$uv = -\frac{1}{2}\beta(u,v)\,e + \gamma(u,v,\cdot)^\dagger = 0.$$

Therefore $N(u) = N(v) = 0$, so N and β are zero on L. By (2.2.2), this also implies $uv = 0$ for all $u,v\in L$. □

Finally, it will be convenient to use certain bases for C and V. We need a well-known lemma:

LEMMA 2.2.4 ([**Sp-Ve**, (1.6.3)]). *There are elements $a, b, c \in C$ such that*

$$e, a, b, ab, c, ac, bc, (ab)c$$

forms an orthogonal basis for C. Such a triple is called a ***basic triple*** *for C.*

In fact, given any $a \in V = e^{\perp}$ with $N(a) = 1$, we can choose b and c so that a, b, c is an *orthonormal* basic triple; similarly, if a and b are orthonormal vectors generating a quaternion subalgebra, we can find c so that a, b, c is an orthonormal basic triple.

If a, b, c are an orthonormal basic triple, let $\{e_0 = e, e_1, \ldots, e_7\}$ be the corresponding basis (in the same order as in Lemma 2.2.4). This is a **standard orthonormal basis** for C. With respect to the basis $\{e_1, \ldots, e_7\}$ for the imaginary octonions V, we have $\beta(e_p, e_q) = 2\,\delta_{pq}$, and

$$\gamma = 2\,(e^*_{123} + e^*_{257} - e^*_{167} - e^*_{145} - e^*_{246} - e^*_{347} - e^*_{356}), \tag{2.2.5}$$

where $e^*_{pqr} = e^*_p \wedge e^*_q \wedge e^*_r$. (Here e^*_p is the map $e_q \mapsto \delta_{pq}$.)

REMARK 2.2.5. Note that for $p > 0$, $e_p^2 = -e$. This standard orthonormal basis is analogous to the standard basis "$1, i, j, k$" for the quaternions. Conventions for defining the octonionic product in terms of a standard basis vary widely in the literature, though — Coxeter [**Co**, p. 562] calculates 480 possible variations! A choice of convention corresponds to a labelling and orientation of the Fano arrangement of 7 points and 7 lines; the one we use agrees with that of [**Fu-Ha**, p. 363]. (Coincidentally, our choice of γ very nearly agrees with the one used in [**Br**, §2]: there the signs of e^*_{347} and e^*_{356} are positive, and the common factor of 2 is absent.)

We will most often use a different basis. Define

$$\begin{aligned} f_1 &= \tfrac{1}{2}(e_1 + i\,e_2) \\ f_2 &= \tfrac{1}{2}(e_5 + i\,e_6) \\ f_3 &= \tfrac{1}{2}(e_4 + i\,e_7) \\ f_4 &= i\,e_3 \\ f_5 &= -\tfrac{1}{2}(e_4 - i\,e_7) \\ f_6 &= -\tfrac{1}{2}(e_5 - i\,e_6) \\ f_7 &= -\tfrac{1}{2}(e_1 - i\,e_2), \end{aligned} \tag{2.2.6}$$

and call this the **standard γ-isotropic basis** for V. (Here i is a fixed square root of -1 in k.) With respect to this basis, the bilinear form is given by

$$\begin{aligned} \beta(f_p, f_{8-q}) &= -\delta_{pq}, \text{ for } p \neq 4 \text{ or } q \neq 4; \\ \beta(f_4, f_4) &= -2. \end{aligned} \tag{2.2.7}$$

The trilinear form is given by

$$\gamma = f^*_{147} + f^*_{246} + f^*_{345} - f^*_{237} - f^*_{156}. \tag{2.2.8}$$

(As above, f^*_p denotes $f_q \mapsto \delta_{pq}$.)

Example 2.2.6. We can use the expression (2.2.8) to compute the octonionic product $f_2\, f_3$. By (2.2.2)–(2.2.4), this is

$$\begin{aligned} f_2\, f_3 &= -\frac{1}{2}\beta(f_2, f_3)\, e + \gamma(f_2, f_3, \cdot)^\dagger \\ &= \gamma(f_2, f_3, \cdot)^\dagger. \end{aligned}$$

Since $\gamma(f_2, f_3, f_j) = -\delta_{7,j} = \beta(f_1, f_j)$, we see $\gamma(f_2, f_3, \cdot)^\dagger = f_1$. Therefore $f_2\, f_3 = f_1$.

We use computations in the f basis to prove another characterization of nondegenerate forms.

Lemma 2.2.7. *Let $\gamma : \bigwedge^3 V \to k$ be a trilinear form, and let β_γ be a symmetric bilinear form defined as in* (1.1.2) *(for* $\mathrm{char}(k) \neq 3$*), by composing*

$$(u, v) \mapsto -\frac{1}{3}\gamma(u, \cdot, \cdot) \wedge \gamma(v, \cdot, \cdot) \wedge \gamma$$

with an isomorphism $\bigwedge^7 V^* \cong k$. Then γ is nondegenerate if and only if β_γ is nondegenerate. (In fact, β_γ is also defined if $\mathrm{char}(k) = 3$*, and the same conclusion holds.)*

Proof. Let $U \subset \bigwedge^3 V^*$ be the set of nondegenerate forms, and let $U' \subset \bigwedge^3 V^*$ be the set of forms γ such that β_γ is nondegenerate; we want to show $U = U'$. (By Proposition A.2.2, U is open and dense.)

First suppose γ is nondegenerate. Since U is a $GL(V)$-orbit in $\bigwedge^3 V^*$, we may choose a basis $\{f_j\}$ so that γ has the expression (2.2.8). Computing in this basis, and using $f^*_{1234567}$ to identify $\bigwedge^7 V^*$ with k, we find $\beta_\gamma = \beta$, i.e., $\beta_\gamma(f_p, f_{8-q}) = -\delta_{pq}$ for $p, q \neq 4$, and $\beta_\gamma(f_4, f_4) = -2$. Indeed, we have

$$\begin{aligned} \gamma(f_1, \cdot, \cdot) \wedge \gamma(f_7, \cdot, \cdot) \wedge \gamma &= (f^*_{47} - f^*_{56}) \wedge (f^*_{14} - f^*_{23}) \wedge \gamma \\ &= 3 f^*_{1234567}. \end{aligned}$$

The others are similar. In particular, with this choice of isomorphism $\bigwedge^7 V^* \cong k$, γ and β_γ are compatible forms. (For an arbitrary choice of isomorphism, β_γ is a scalar multiple of a compatible form.)

To see this works in characteristic 3, one can avoid division by 3. Let $V_{\mathbb{Z}}$ be a rank 7 free $\mathbb{Z}$-module, fix a basis $f_1, \ldots, f_7$, and let $\gamma_{\mathbb{Z}} : \bigwedge^3 V_{\mathbb{Z}} \to \mathbb{Z}$ be given by (2.2.8). The same computation shows that

$$\gamma_{\mathbb{Z}}(f_p, \cdot, \cdot) \wedge \gamma_{\mathbb{Z}}(f_{8-q}, \cdot, \cdot) \wedge \gamma_{\mathbb{Z}} = 3\, \delta_{pq}\, f^*_{1234567}$$

for $p, q \neq 4$, and

$$\gamma_{\mathbb{Z}}(f_4, \cdot, \cdot) \wedge \gamma_{\mathbb{Z}}(f_4, \cdot, \cdot) \wedge \gamma_{\mathbb{Z}} = 6\ f^*_{1234567},$$

so one can define β_γ over $\mathbb{Z}$. (For nondegeneracy, one still needs $\mathrm{char}(k) \neq 2$ here.)

For the converse, note that the terms in the compatibility relation (1.1.1) make sense for all γ in U', since here $\gamma(u, v, \cdot)^\dagger$ is well-defined. We have seen that the relation holds on the dense open subset $U \subset U'$, so it must hold on all of U'. Therefore every γ in U' has a compatible bilinear form, i.e., γ is in U. $\square$

The following two lemmas prove Proposition 1.1.2:

LEMMA 2.2.8. *If $u \in V$ is a nonzero isotropic vector, then*

$$\begin{aligned} E_u &= \{v \in V \mid uv = 0\} \\ &= \{v \in V \mid \gamma(u, v, \cdot) \equiv 0\} \end{aligned}$$

is a three-dimensional β-isotropic subspace.

PROOF. By definition, E_u consists of zero-divisors, so it is β-isotropic by (2.1.5). Since β is nondegenerate on V, we know $\dim E_u \leq 3$.

In fact, it is enough to observe that $G = \mathrm{Aut}(C)$ acts transitively on the set of isotropic vectors (up to scalar); this follows from Proposition A.4.1(a). Thus for any u, we can find $g \in G$ such that $g \cdot u = \lambda f_1$ for some $\lambda \neq 0$. Clearly $g \cdot E_u = E_{g \cdot u} = E_{f_1}$, and one checks that $f_1 f_2 = f_1 f_3 = 0$. $\square$

LEMMA 2.2.9. *Let $u \in V$ be a nonzero isotropic vector, and let $v, w \in E_u$ be such that $\{u, v, w\}$ is a basis. Then $vw = \lambda u$ for some nonzero $\lambda \in k$.*

PROOF. First note that $vw = -wv$, since $-vw - wv = \beta(v, w)e = 0$. If $\{u, v', w'\}$ is another basis, with $v' = a_1 u + a_2 v + a_3 w$ and $w' = b_1 u + b_2 v + b_3 w$, then $a_2 b_3 - a_3 b_2 \neq 0$, so

$$v'w' = (a_2 b_3) vw + (a_3 b_2) wv = (a_2 b_3 - a_3 b_2) vw$$

is a nonzero multiple of vw. Now it suffices to check this for the standard γ-isotropic basis, and indeed, we computed $f_2 f_3 = f_1$ in Example 2.2.6. $\square$

COROLLARY 2.2.10. *Let $V = L_1 \oplus \cdots \oplus L_7$ be a splitting into one-dimensional subspaces such that L_1 is γ-isotropic, and $L_1 \oplus L_2 \oplus L_3 = E_u$ for a generator $u \in L_1$. Then the map $V \otimes V \to V^* \cong V$ induced by γ restricts to a G-equivariant isomorphism $L_2 \otimes L_3 \xrightarrow{\sim} L_1$.*

Finally, the following lemma is verified by a straightforward computation:

LEMMA 2.2.11. *Let $T = (k^*)^2$ act on V via the matrix*

$$\mathrm{diag}(z_1, z_2, z_1 z_2^{-1}, 1, z_1^{-1} z_2, z_2^{-1}, z_1^{-1})$$

(in the f-basis). Then T preserves the forms β and γ of (2.2.7) *and* (2.2.8)*.*

The corresponding weights for this torus action are $\{t_1, t_2, t_1 - t_2, 0, t_2 - t_1, -t_2, -t_1\}$.

2.3. Octonion bundles

Let X be a variety over k. The notion of composition algebra can be globalized:

DEFINITION 2.3.1. A **composition algebra bundle** over X is a vector bundle $C \to X$, equipped with a nondegenerate quadratic norm $N : C \to k_X$, a multiplication $m : C \otimes C \to C$, and an identity section $e : k_X \to C$, such that N respects composition. (Equivalently, for each $x \in X$, the fiber $C(x)$ is a composition algebra over k.)

Since $\mathrm{char}(k) \neq 2$, there is a corresponding nondegenerate bilinear form β' on C. We will also allow composition algebras whose norm takes values in a line bundle $M^{\otimes 2}$; here the multiplication is $C \otimes C \xrightarrow{m} C \otimes M$, and the identity is $M \xrightarrow{e} C$. Here a little care is required in the definition. The composition $C \otimes M \xrightarrow{id \otimes e} C \otimes C \xrightarrow{m} C \otimes M$ should be the identity, and the other composition $(m \circ (e \otimes id))$ should be the canonical isomorphism. The compatibility between m and N is encoded in the commutativity of the following diagram:

$$\begin{tikzcd} C \otimes C \arrow[rr, "m"] \arrow[dr, "N \otimes N"'] & & C \otimes M \arrow[dl, "N \otimes (N \circ e)"] \\ & M^{\otimes 4}. & \end{tikzcd}$$

The norm of e is the quadratic map $M \to M^{\otimes 2}$ corresponding to $M^{\otimes 2} \xrightarrow{\beta'} M^{\otimes 2}$. Replacing C with $\widetilde{C} = C \otimes M^*$, one obtains a composition algebra whose norm takes values in the trivial bundle.

Many of the properties of composition algebras discussed above have straightforward generalizations to bundles; we mention a few without giving proofs.

Using β' to identify C with $C^* \otimes M^{\otimes 2}$, the multiplication map corresponds to a trilinear form $\gamma' : C \otimes C \otimes C \to M^{\otimes 3}$. The **imaginary subbundle** V is the orthogonal complement to e in C, so $C = M \oplus V$. The bilinear form β' restricts to a nondegenerate form β on V, and γ' restricts to an alternating form $\gamma : \bigwedge^3 V \to M^{\otimes 3}$. As before, the multiplication on $M \oplus V$ can be recovered from the forms β and γ on V, and there is an analogue of Proposition 2.2.1.

The analogues of Proposition 1.1.2 and Corollary 2.2.10 can be proved using octonion bundles and reducing to the local case:

PROPOSITION 2.3.2. *Let $\gamma : \bigwedge^3 V \to M^{\otimes 3}$ and $\beta : \mathrm{Sym}^2 V \to M^{\otimes 2}$ be (locally) compatible forms. Let $F_1 \subset V$ be a γ-isotropic line bundle, and let $\varphi : V \to F_1^* \otimes V^* \otimes M^{\otimes 3}$ be the map defined by γ. Then the bundle*

$$E_{F_1} = \ker(\varphi)$$

has rank 3 and is β-isotropic.

Proposition 2.3.3. *Let V be as in Proposition 2.3.2, and suppose there is a splitting $V = L_1 \oplus \cdots \oplus L_7$ into line bundles such that L_1 is γ-isotropic, and $L_1 \oplus L_2 \oplus L_3 = E_{L_1}$. Then the map $V \otimes V \to V^* \cong V \otimes M$ induced by γ and β restricts to an isomorphism $L_2 \otimes L_3 \xrightarrow{\sim} L_1 \otimes M$.*

Remark 2.3.4. Composition algebras may defined over an arbitrary base scheme X; in fact, as with Azumaya algebras, one is mainly interested in cases where X is defined over a non-algebraically closed field or a Dedekind ring. Petersson has classified such composition algebra bundles in the case where X is a curve of genus zero [**Pe**]. Since then, some work has been done over other one-dimensional bases, but the theory remains largely undeveloped.

2.4. Another construction

We shall need a construction of octonion algebras which works in bundles. This is essentially equivalent to the Cayley-Dickson doubling construction; see also [**Pe**] or [**Mu**].

First we fix some notation. For any vector bundle E, let $Tr : End(E) = E^* \otimes E \to \mathcal{O}_X$ be the canonical contraction map, and let $End^0(E) = \ker(Tr) \subset End(E)$ be the subbundle of trace-zero endomorphisms. Let $e : \mathcal{O}_X \to End(E)$ be the identity section. Thus the composition $Tr \circ e : \mathcal{O}_X \to \mathcal{O}_X$ is multiplication by $\mathrm{rk}(E)$. Also, the *conjugation* map $End(E) \to End(E)$ is given by $e \circ Tr - id$. (Here id is the identity morphism, as opposed to the identity section e.) Conjugation is an involution; locally, it is $\xi \mapsto \overline{\xi} := Tr(\xi)e - \xi$.

The main result of this section is a G_2 analogue of the well-known fact that for any vector bundle E, the direct sum $E \oplus E^*$ carries canonical symplectic (type C) and symmetric (type D) forms; see e.g. [**Fu-Pr**, p. 71].

Proposition 2.4.1. *Let E be a rank 2 vector bundle on a variety X. Then $C = E \oplus End(E) \oplus E^*$ has a canonical octonion bundle structure, with identity section $e : \mathcal{O}_X \to End(E) \subset C$. In particular,*

$$V = E \oplus End^0(E) \oplus E^* \subset C$$

has a canonical nondegenerate alternating trilinear form $\gamma : \bigwedge^3 V \to \mathcal{O}_X$, with a compatible bilinear form β. The subbundle $E = E \oplus 0 \oplus 0 \subset V$ is γ-isotropic.

Proof. We need to define the norm $N : C \to \mathcal{O}_X$ and multiplication $m : C \otimes C \to C$, for $C = E \oplus End(E) \oplus E^*$, and check that they are compatible.

The norm on C corresponds to the bilinear form β' defined locally by

$$\beta'(x \oplus \xi \oplus f,\, y \oplus \eta \oplus g) = Tr(\xi)Tr(\eta) - Tr(\xi\eta) - f(y) - g(x).$$

(This can also be expressed in terms of natural contraction maps.) It is clear that β' is nondegenerate. Thus

$$N(x \oplus \xi \oplus f) = \det(\xi) - f(x)$$

is a nondegenerate quadratic norm on C.

The multiplication is given by

$$(x \oplus \xi \oplus f) \cdot (y \oplus \eta \oplus g) \quad = \quad (\eta x + \overline{\xi} y) \oplus (\overline{g \otimes x} + \xi\eta + f \otimes y) \oplus (g\xi + f\overline{\eta}).$$

Noting that $\overline{e} = e$, it is easy to see that e (the identity for $End(E)$) acts as a multiplicative identity for C. Moreover, the multiplication restricts to zero on $E \oplus 0 \oplus 0 \subset C$.

To verify the multiplicativity of the norm, we compute:

$$\begin{aligned}
N((x \oplus \xi \oplus f) \cdot (y \oplus \eta \oplus g)) &= \det(\overline{g \otimes x} + \xi\eta + f \otimes y) - (g\xi + f\overline{\eta})(\eta x + \overline{\xi} y) \\
&= \det(\xi\eta) + \beta'(\overline{g \otimes x}, \xi\eta) + \beta'(\overline{g \otimes x}, f \otimes y) \\
&\quad + \beta'(\xi\eta, f \otimes y) - (g\xi\eta x + g\xi\overline{\xi} y + f\overline{\eta}\eta x + f\overline{\eta}\overline{\xi} y) \\
&= \det(\xi)\det(\eta) + \beta'(g(x)e, \xi\eta) - \beta'(g \otimes x, \xi\eta) \\
&\quad + \beta'(g(x)e, f \otimes y) - \beta'(g \otimes x, f \otimes y) + \beta'(\xi\eta, f \otimes y) \\
&\quad - g\xi\eta x - \det(\xi) g(y) - f(x)\det(\eta) - f\overline{\xi\eta} y \\
&= \det(\xi)\det(\eta) + g(x)Tr(\xi\eta) - g(x)Tr(\xi\eta) + g\xi\eta x \\
&\quad + g(x)f(y) - g(x)f(y) + g(y)f(x) + Tr(\xi\eta)f(y) \\
&\quad - f\xi\eta y - g\xi\eta x - \det(\xi)g(y) - f(x)\det(\eta) - f\overline{\xi\eta} y \\
&= \det(\xi)\det(\eta) - \det(\xi)g(y) - f(x)\det(\eta) + f(x)g(y) \\
&\quad + Tr(\xi\eta)f(y) - f\xi\eta y - f(Tr(\xi\eta)e - \xi\eta)y \\
&= \det(\xi)\det(\eta) - \det(\xi)g(y) - f(x)\det(\eta) + f(x)g(y) \\
&= N(x \oplus \xi \oplus f)\, N(y \oplus \eta \oplus g).
\end{aligned}$$

Thus we have defined an octonion algebra structure on C. Compatible forms γ and β on $V = E \oplus End^0(E) \oplus E^*$ are obtained by restricting the multiplication and norm.

Since the multiplication is zero on $E = E \oplus 0 \oplus 0$, it follows that $E \subset V$ is γ-isotropic. $\square$

It will be convenient to use a basis adapted to this construction, in the case where X is a point, so E is a 2-dimensional vector space. Let v_1, v_2 be a basis for E, and extend to a basis for $C = E \oplus \mathrm{End}(E) \oplus E^*$ by setting

$$\begin{aligned}
v_3 &= v_2^* \otimes v_1 \\
v_4 &= v_1^* \otimes v_1 \\
v_5 &= v_2^* \otimes v_2 \\
v_6 &= v_1^* \otimes v_2 \\
v_7 &= v_2^* \\
v_8 &= v_1^*.
\end{aligned} \tag{2.4.1}$$

Thus the identity element is $e = v_4 + v_5$, and the relation to the standard γ-isotropic basis (2.2.6) is given by

$$
\begin{aligned}
v_1 &= f_1 \\
v_2 &= f_2 \\
v_3 &= f_3 \\
v_4 &= \tfrac{1}{2}(e + f_4) \\
v_5 &= \tfrac{1}{2}(e - f_4) \\
v_6 &= f_5 \\
v_7 &= f_6 \\
v_8 &= f_7.
\end{aligned}
\tag{2.4.2}
$$

With respect to this basis, the symmetric bilinear form β' is given by

$$
\begin{aligned}
\beta'(v_p, v_{9-q}) &= -\delta_{pq}, \text{ for } p, q \neq 4, 5; \\
\beta'(v_4, v_5) &= 1.
\end{aligned}
\tag{2.4.3}
$$

The torus $T = (k^*)^2$ acts on C in this basis by the matrix

$$\mathrm{diag}(z_1, z_2, z_1 z_2^{-1}, 1, 1, z_1^{-1} z_2, z_2^{-1}, z_1^{-1}),$$

with weights $\{t_1, t_2, t_1 - t_2, 0, 0, -t_1 + t_2, -t_2, -t_1\}$. This is induced from the standard action on $E = \langle v_1, v_2 \rangle$.

Remark 2.4.2. This construction yields a natural embedding $GL(E) \hookrightarrow \mathrm{Aut}(C)$. In fact, the subgroup of $G = \mathrm{Aut}(C)$ stabilizing E is parabolic (Proposition A.4.1), and $GL(E) \cong GL_2$ is a Levi subgroup.

Remark 2.4.3. Let R be any commutative ring. Proposition 2.4.1 clearly also holds when E is a locally free R-module of rank 2, with the same construction. (In particular, this constructs (split) octonion bundles over any scheme, or more generally, any locally ringed space. Compare [**Pe**].)

CHAPTER 3

Isotropic flags and flag bundles

In this chapter, we describe some basic properties of the variety Fl_γ defined in §1.1.2.

3.1. Topology of G_2 flags

There are two "γ-isotropic Grassmannians" parametrizing γ-isotropic subspaces of dimensions 1 or 2, which we write as $\mathcal{Q}$ or $\mathcal{G}$, respectively; thus Fl_γ embeds in $\mathcal{Q} \times \mathcal{G}$. Since γ-isotropic vectors are just those v such that $\beta(v, v) = 0$, $\mathcal{Q}$ is the smooth 5-dimensional quadric hypersurface in $\mathbb{P}(V)$.

PROPOSITION 3.1.1. *The γ-isotropic flag variety is a smooth,* 6*-dimensional projective variety. Moreover, both projections $Fl_\gamma \to \mathcal{Q}$ and $Fl_\gamma \to \mathcal{G}$ are $\mathbb{P}^1$-bundles.*

PROOF. The quadric $\mathcal{Q}$ comes with a tautological line bundle $S_1 \subset V_{\mathcal{Q}}$. By Proposition 1.1.2, the form γ also equips $\mathcal{Q}$ with a rank-3 bundle $S_3 \subset V_{\mathcal{Q}}$, with fiber $S_3([u]) = E_u$, the space swept out by all γ-isotropic 2-spaces containing u. Thus $S_1 \subset S_3$, and from the definitions we have $Fl_\gamma(V) = \mathbb{P}(S_3/S_1) \to \mathcal{Q}$. (We use the convention that $\mathbb{P}(E)$ parametrizes lines in the vector bundle E.)

Similarly, if S_2 is the tautological bundle on $\mathcal{G}$, we have $Fl_\gamma(V) = \mathbb{P}(S_2) \to \mathcal{G}$. This also shows that $\mathcal{G}$ is smooth of dimension 5. $\square$

REMARK 3.1.2. The definition of $Fl_\gamma(V)$ can be reformulated as follows. Let $Fl = Fl(1, 2; V)$ be the two-step partial flag variety. The nondegenerate form γ is also a section of the trivial vector bundle $\bigwedge^3 V^*$ on Fl. By restriction it gives a section of the rank 5 vector bundle $\bigwedge^2 S_2^* \otimes Q_5^*$, where $S_1 \subset S_2 \subset V$ is the tautological flag on Fl and $Q_5 = V/S_2$. Then $Fl_\gamma \subset Fl$ is defined by the vanishing of this section.

REMARK 3.1.3. Projectively, Fl_γ parametrizes data $(p \in \ell)$, where ℓ is a γ-isotropic line in $\mathcal{Q}$, and $p \in \ell$ is a point. Thus Proposition 1.1.2 says that the union of such ℓ through a fixed p is a $\mathbb{P}^2$ in $\mathcal{Q}$, and conversely, given such a $\mathbb{P}^2$ one can recover p (as the intersection of any two γ-isotropic lines in the $\mathbb{P}^2$).

This suggests another description of Fl_γ. Consider $\mathcal{Q}$ with its bundles $S_1 \subset S_3$, and let $\mathbf{Fl}(S_3) \to \mathcal{Q}$ be the bundle of (all) flags in S_3. Write S_1 and S_3 also for their pullbacks to $\mathbf{Fl}(S_3)$, and let $U_1 \subset U_2 \subset U_3 = S_3$ be the tautological bundles on $\mathbf{Fl}(S_3)$.

PROPOSITION 3.1.4. *In the notation of* [**Fu1**], Fl_γ *is the Schubert variety* $\mathbf{\Omega}_{231}$ *in the flag bundle* $\mathbf{Fl}(S_3)$*:*

$$Fl_\gamma = \mathbf{\Omega}_{231} = \{x \mid \dim(S_1(x) \cap U_1(x)) \geq 1\} \subset \mathbf{Fl}(S_3).$$

3.1.1. Fixed points. Let $\{f_1, f_2, \ldots, f_7\}$ be the standard γ-isotropic basis for V, and let $T = (k^*)^2$ act as in Lemma 2.2.11, via the matrix $\mathrm{diag}(z_1, z_2, z_1 z_2^{-1}, 1, z_1^{-1} z_2, z_2^{-1}, z_1^{-1})$. Write $e(i\,j)$ for the two-step flag $\langle f_i \rangle \subset \langle f_i, f_j \rangle$.

PROPOSITION 3.1.5. *This action of* T *defines an action on* $Fl_\gamma(V)$*, with* 12 *fixed points:*

$$\begin{gathered} e(1\,2),\ e(1\,3),\ e(2\,1),\ e(2\,5),\ e(3\,1),\ e(3\,6), \\ e(5\,2),\ e(5\,7),\ e(6\,3),\ e(6\,7),\ e(7\,5),\ e(7\,6). \end{gathered}$$

PROOF. Since T preserves β, it acts on $\mathcal{Q}$, fixing the 6 points $[f_1]$, $[f_2]$, $[f_3]$, $[f_5]$, $[f_6]$, $[f_7]$. Since T preserves γ, it acts on Fl_γ, and the projection $Fl_\gamma \to \mathcal{Q}$ is T-equivariant. The T-fixed points of Fl_γ lie in the fibers over the fixed points of $\mathcal{Q}$. Since each of these 6 fibers is a $\mathbb{P}^1$ with nontrivial T-action, there must be $2 \cdot 6 = 12$ fixed points.

To see the fixed points are as claimed, note that the bundle S_3 on $\mathcal{Q}$ is equivariant, and the fibers $S_3(x) = E_x$ at each of the fixed points are as follows:

$$\begin{aligned} E_{f_1} &= \langle f_1, f_2, f_3 \rangle \\ E_{f_2} &= \langle f_2, f_1, f_5 \rangle \\ E_{f_3} &= \langle f_3, f_1, f_6 \rangle \\ E_{f_5} &= \langle f_5, f_2, f_7 \rangle \\ E_{f_6} &= \langle f_6, f_3, f_7 \rangle \\ E_{f_7} &= \langle f_7, f_5, f_6 \rangle. \end{aligned}$$

Indeed, one simply checks that in each triple, the (octonionic) product of the first vector with either the second or the third is zero. (Alternatively, one can compute directly using the form (2.2.8).) Now the T-fixed lines in $S_3([f_i])/S_1([f_i])$ are $[f_j]$, where f_j is the second or third vector in the triple beginning with f_i. Thus the 12 points are $e(i\,j)$, where f_i is the first vector and f_j is the second or third vector in one of the above triples. $\square$

In general, the T-fixed points of a flag variety are indexed by the corresponding Weyl group W, which for type G_2 is the dihedral group with 12 elements. We will write elements as $w = w_1\, w_2$, for w_1 and w_2 such that $e(w_1\, w_2)$ is a T-fixed point, as in Proposition 3.1.5. We fix two simple reflections generating W, $s = 2\,1$ and $t = 1\,3$. See §A.3 for more details on the Weyl group and its embedding in S_7.

3.1.2. Schubert varieties. Fix a (complete) γ-isotropic flag $F_\bullet$ in V. Each T-fixed point is the center of a **Schubert cell**, which is defined by

$$X^o_w = \{E_\bullet \in Fl_\gamma \mid \dim(F_p \cap E_q) = r_w(q,p) \text{ for } 1 \le q \le 2,\, 1 \le p \le 7\},$$

where

$$r_w(q,p) = \#(i \le q \mid w_i \le p),$$

just as in the classical types. Also as in classical types, these can be parametrized by matrices, where E_i is the span of the first i rows. For example, the big cell is

$$X^o_{7\,6} = \begin{pmatrix} X & a & b & c & d & e & 1 \\ Y & Z & S & T & f & 1 & 0 \end{pmatrix} \cong \mathbb{A}^6,$$

where lowercase variables are free, and X, Y, Z, S, T are given by

$$\begin{aligned} X &= -ae - bd - c^2 \\ Y &= -a - bf + cd - cef \\ Z &= -cf - d^2 + def \\ S &= c + de - e^2 f \\ T &= -d + ef. \end{aligned}$$

(These equations can be obtained by octonionic multiplication; considering the two row vectors as imaginary octonions, the condition is that their product be zero. In fact, X, Y, Z are already determined by β-isotropicity.) Parametrizations of the other 11 cells are given in Appendix D.1.

The **Schubert varieties** X_w are the closures of the Schubert cells; equivalently,

$$X_w = \{E_\bullet \in Fl_\gamma \mid \dim(F_p \cap E_q) \ge r_w(q,p) \text{ for } 1 \le q \le 2,\, 1 \le p \le 7\}.$$

From the parametrizations of cells, we see $\dim X_w = \ell(w)$. To get Schubert varieties with codimension $\ell(w)$, define

$$\Omega_w = X_{w\,w_0}.$$

These can also be described using the tautological quotient bundles:

$$\Omega_w = \{x \in Fl_\gamma \mid \operatorname{rk}(F_p(x) \to Q_q(x)) \le r_w(q,p)\}.$$

Schubert varieties in $\mathcal{Q}$ and $\mathcal{G}$ are defined by the same conditions. (Note that w and $w\,s$ define the same varieties in $\mathcal{G}$, and w and $w\,t$ define the same variety in $\mathcal{Q}$. Write $\overline{w}$ for the corresponding equivalence class.) With the exception of $X_{1\,2}$, all Schubert varieties in Fl_γ are inverse images of Schubert varieties in $\mathcal{Q}$ or $\mathcal{G}$:

Proposition 3.1.6. *Let $p : Fl_\gamma \to \mathcal{Q}$ and $q : Fl_\gamma \to \mathcal{G}$ be the projections. Then $X_w = p^{-1} X_{\overline{w}}$ if $w_1 < w_2$ (except when $w = 1\,2$), and $X_w = q^{-1} X_{\overline{w}}$ if $w_1 > w_2$.*

The proof is immediate from the definitions. For instance, $X_{\overline{tst}} = X_{\overline{36}}$ is a $\mathbb{P}^2$ in $\mathcal{Q}$: it parametrizes all 1-dimensional subspaces of a fixed isotropic 3-space. Its inverse image in Fl_γ is $p^{-1}X_{\overline{tst}} = X_{tst} = \Omega_{sts}$.

3.2. Cohomology of flag bundles

3.2.1. Compatible forms on bundles. Let V be a rank 7 vector bundle on a variety X, equipped with a nondegenerate form $\gamma : \bigwedge^3 V \to L$, and let $B_\gamma : \mathrm{Sym}^2 V \to \det V^* \otimes L^{\otimes 3}$ be the Bryant form (§1.1.1, §1.1.3). Assume there is a line bundle M such that

$$\det V^* \otimes L^{\otimes 3} \cong M^{\otimes 2}. \tag{3.2.1}$$

(For example, this holds if V has a maximal B_γ-isotropic subbundle F, for then we can take $M = F^\perp/F$. There exist Zariski-locally trivial bundles V without this property, though — see [**Ed-Gr1**, p. 293].)

Lemma 3.2.1. *In this setup, $L \cong M^{\otimes 3} \otimes T$, for some line bundle T such that $T^{\otimes 3}$ is trivial. If L has a cube root, then T is trivial and $M \cong \det V \otimes (L^*)^{\otimes 2}$.*

Proof. In general, if V is a bundle of rank r with a quadratic form with values in $M^{\otimes 2}$, we have $\det V \cong M^{\otimes r}$. (One can use a splitting principle to assume V is a sum of line bundles $L_1, \ldots, L_N$, with $L_{N+1-i} = L_i^* \otimes M^{\otimes 2}$.) Thus in our case, $\det V \cong M^{\otimes 7}$. From (3.2.1), we have $L^{\otimes 3} \cong M^{\otimes 9}$, and the first statement follows.

For the second statement, first assume L is trivial. In this case, the structure group of V is contained in $\mu_3 \times SL_7$ (see Proposition A.2.2), and it follows that $(\det V)^{\otimes 3}$ is trivial. Since $\det V^* \cong M^{\otimes 2}$, this implies $M^{\otimes 6}$ is trivial. On the other hand, $M^{\otimes 6} = (M^{\otimes 3})^{\otimes 2} \cong (T^*)^{\otimes 2} \cong T^{\otimes 4} \cong T$, so T is trivial and $M \cong \det V$.

Finally, suppose $L \cong K^{\otimes 3}$ for some line bundle K. Replacing V with $\widetilde{V} = V \otimes K^*$, we obtain a nondegenerate form $\widetilde{\gamma} : \bigwedge^3 \widetilde{V} \to k_X$. By the previous paragraph, $(\det \widetilde{V})^{\otimes 3} = (\det V)^{\otimes 3} \otimes (K^*)^{\otimes 21}$ is trivial, so $(\det V)^{\otimes 3} \cong L^{\otimes 7}$. On the other hand, using (3.2.1) we have $(\det V)^{\otimes 3} \cong L^{\otimes 6} \otimes M^{\otimes 3}$, so $L \cong M^{\otimes 3}$, and $\det V \otimes (L^*)^{\otimes 2} \cong M$. □

3.2.2. A splitting principle. From now on, we will assume V has a maximal B_γ-isotropic subbundle $F = F_3 \subset V$. We also assume L has a cube root on X, so $L \cong M^{\otimes 3}$. (By a theorem of Totaro, one can always assume this so long as 3-torsion is ignored in Chow groups (or cohomology); see [**Fu2**]. In the case at hand, Lemma 3.2.1 gives a direct reason.)

In this context, the relevant version of the splitting principle is the following:

Lemma 3.2.2. *Assume V is equipped with a nondegenerate trilinear form $\gamma : \bigwedge^3 V \to M^{\otimes 3}$. There is a map $f : Z \to X$ such that $f^* : H^*X \to H^*Z$ is injective, and $f^*V \cong L_1 \oplus L_2 \oplus \cdots \oplus L_7$, with $E_i = L_1 \oplus \cdots \oplus L_i$ forming a complete γ-isotropic flag in f^*V.*

PROOF. Set $Z' = \mathbf{Fl}_\gamma(V)$, obtaining the tautological filtration $S_\bullet$ of V. Since V has a maximal isotropic subbundle, it (with its quadratic form) is Zariski-locally trivial, and therefore $Z' \to X$ is also a Zariski-locally trivial bundle. By [**Ed-Gr1**, Lemma 3], the map $H^*X \to H^*Z'$ is injective. Then argue as in [**Fu2**, p. 246] to find an affine bundle $Z \to Z'$ where the tautological filtration splits. (Over $\mathbb{C}$, one can simply take $Z = \mathbf{Fl}_\gamma(V)$, which is analytically-locally trivial over X, and use a Hermitian metric to split the tautological filtration.) $\square$

Given such a splitting, we can use β to identify L_{8-i} with $L_i^* \otimes M^{\otimes 2}$, and Proposition 2.3.3 implies $L_3 \cong L_1 \otimes L_2^* \otimes M$. Thus

$$\begin{aligned} V \;\cong\; & L_1 \oplus L_2 \oplus (L_1 \otimes L_2^* \otimes M) \oplus M \\ & \oplus (L_1^* \otimes L_2 \otimes M) \oplus (L_2^* \otimes M^{\otimes 2}) \oplus (L_1^* \otimes M^{\otimes 2}). \end{aligned} \tag{3.2.2}$$

A little more concisely, if $F_1 \subset F_2 \subset V$ is a γ-isotropic flag of subbundles, we have

$$V \cong F_2 \oplus (F_1 \otimes (F_2/F_1)^* \otimes M) \oplus M \oplus (F_1^* \otimes (F_2/F_1) \otimes M) \oplus (F_2^* \otimes M^{\otimes 2}).$$

Since V is recovered from the data of L_1, L_2, and M, the universal base for V is $(BGL_1)^3$. This space has no torsion in cohomology; it follows that we may deduce integral formulas using rational coefficients.

If M is trivial, (3.2.2) shows

$$c(V) = (1 - y_1^2)(1 - y_2^2)(1 - (y_1 - y_2)^2), \tag{3.2.3}$$

where $y_i = c_1(L_i)$.

3.2.3. Chern classes. Assume the line bundle M is trivial, and let $F_1 \subset F_2 \subset F_3 \subset V$ be a γ-isotropic flag in V. It follows from (3.2.2) that

$$c_1(F_3) = 2\, c_1(F_1).$$

Let $\mathcal{Q}(V) \to X$ be the quadric bundle, with its tautological bundles $S_1 \subset S_3 \subset V$. Set $x_1 = -c_1(S_1)$ and $\alpha = [\mathbb{P}(F_3)]$ in $H^*\mathcal{Q}(V)$. The classes 1, x_1, x_1^2, α, $x_1\,\alpha$, $x_1^2\,\alpha$ form a basis for $H^*\mathcal{Q}(V)$ over H^*X; see Appendix E.

LEMMA 3.2.3. *We have*

$$\begin{aligned} c_1(S_3) &= -2\,x_1, \qquad \textit{and} \\ c_2(S_3) &= 2\,x_1^2 + c_2(F_3) - 2\,c_1(F_1)^2. \end{aligned}$$

PROOF. The expression for $c_1(S_3)$ follows from (3.2.2). Since M is assumed trivial, we have $V/F_3^\perp \cong F_3^*$ and $V/S_3^\perp \cong S_3^*$, so $c(V) = c(F_3) \cdot c(F_3^*) = c(S_3) \cdot c(S_3^*)$. In particular,

$$\begin{aligned} c_2(V) &= 2\,c_2(F_3) - c_1(F_3)^2 = 2\,c_2(S_3) - c_1(S_3)^2 \\ &= 2(c_2(F_3) - 2\,c_1(F_1)^2) = 2(c_2(S_3) - 2\,x_1^2). \end{aligned}$$

Up to 2-torsion, then, the formula for $c_2(S_3)$ holds. Since the classifying space for this setup is $BGL_1 \times BGL_1$, and there is no torsion in its cohomology, it follows that the formula also holds with integer coefficients. $\square$

3.2.4. Presentations. Using the fact that $\mathbf{Fl}_\gamma(V)$ is a $\mathbb{P}^1$-bundle over a quadric bundle, we can give a presentation of its integral cohomology. First recall the presentation for $H^*\mathcal{Q}(V)$ (Theorem E.1). We continue to assume M is trivial, and hence also $\det V$. Fix $F_1 \subset F_3 \subset V$ as before, and let $S_1 \subset S_3 \subset V$ be the tautological bundles on $\mathcal{Q}(V)$. Let $x_1 = -c_1(S_1)$ and $\alpha = [\mathbb{P}(F_3)]$ in $H^*\mathcal{Q}(V)$. Then

$$H^*(\mathcal{Q}(V), \mathbb{Z}) = (H^*X)[x_1, \alpha]/I,$$

where I is generated by

$$\begin{aligned} 2\alpha &= x_1^3 - c_1(F_3)\, x_1^2 + c_2(F_3)\, x_1 - c_3(F_3), \\ \alpha^2 &= (c_3(V/F_3) + c_1(V/F_3)\, x_1^2)\, \alpha. \end{aligned}$$

THEOREM 3.2.4. *With notation as above, we have $\mathbf{Fl}_\gamma(V) = \mathbb{P}(S_3/S_1) \to \mathcal{Q}(V) \to X$. Let $x_2 = -c_1(S_2/S_1)$ be the hyperplane class for this $\mathbb{P}^1$-bundle. Then*

$$H^*(\mathbf{Fl}_\gamma(V), \mathbb{Z}) = (H^*X)[x_1, x_2, \alpha]/J,$$

where J is generated by the three relations

$$2\alpha = x_1^3 - c_1(F_3)\, x_1^2 + c_2(F_3)\, x_1 - c_3(F_3), \tag{3.2.4}$$

$$\alpha^2 = (c_3(V/F_3) + c_1(V/F_3)\, x_1^2)\, \alpha, \tag{3.2.5}$$

$$x_1^2 + x_2^2 - x_1 x_2 = 2\, c_1(F_1)^2 - c_2(F_3). \tag{3.2.6}$$

In fact, α is the Schubert class $[\mathbf{\Omega}_{sts}]$, defined in §4.1 below.

PROOF. Since $\mathbf{Fl}_\gamma(V) = \mathbb{P}(S_3/S_1) \to \mathcal{Q}(V)$, we have

$$H^*\mathbf{Fl}_\gamma = (H^*\mathcal{Q})[x_2]/(x_2^2 + c_1(S_3/S_1)\, x_2 + c_2(S_3/S_1)).$$

One easily checks $c_1(S_3/S_1) = -x_1$, and

$$c_2(S_3/S_1) = c_2(S_3) - x_1^2 = x_1^2 + c_2(F_3) - 2\, c_1(F_1)^2$$

by Lemma 3.2.3. This gives the third relation, and the first two relations come from the relations on $H^*\mathcal{Q}$.

Finally, it is not hard to see that the 12 elements

$$1,\, x_1, x_1^2,\, \alpha,\, x_1\, \alpha,\, x_1^2\, \alpha,\, x_2,\, x_1\, x_2,\, x_1^2\, x_2,\, x_2\, \alpha,\, x_1\, x_2\, \alpha,\, x_1^2\, x_2\, \alpha$$

form a basis for the ring on the RHS over H^*X, and we know they form a basis for $H^*\mathbf{Fl}_\gamma$ over H^*X. $\square$

REMARK 3.2.5. To obtain a presentation for $H_T^*(Fl_\gamma, \mathbb{Z})$, set $\alpha = [\Omega_{sts}]^T$, $x_i = -c_1^T(S_i/S_{i-1})$, $c_i(F_3) = (-1)^i c_i(V/F_3) = e_i(t_1, t_2, t_1 - t_2)$, and $c_1(F_1) = t_1$.

If we take coefficients in $\mathbb{Z}[\frac{1}{2}]$, the cohomology ring has a simpler presentation similar to that for classical groups:

PROPOSITION 3.2.6. *Suppose V has a splitting as in* (3.2.2)*, with M trivial. Let $\Lambda = H^*X$. Then $H^*(\mathbf{Fl}_\gamma(V), \mathbb{Z}[\frac{1}{2}]) \cong \Lambda[x_1, x_2]/(r_2, r_4, r_6)$, where*

$$r_{2i} \quad = \quad e_i(x_1^2, x_2^2, (x_1 - x_2)^2) - e_i(y_1^2, y_2^2, (y_1 - y_2)^2).$$

PROOF. The relations must hold, by (3.2.3). Monomials in x_1 and x_2 are global classes on $\mathbf{Fl}_\gamma$ that restrict to give a basis for the cohomology of each fiber, so the claim follows from the Leray-Hirsch theorem. $\square$

Taking X to be a point, these presentations specialize to give well-known presentations of H^*Fl_γ (cf. [**Bo-Sa**]):

COROLLARY 3.2.7. *Let Fl_γ be the γ-isotropic flag variety, and let $p : Fl_\gamma \to \mathcal{Q}$ be the projection to the quadric. Set $\alpha = [\Omega_{sts}] \in H^*(Fl_\gamma, \mathbb{Z})$. Then we have*

$$H^*(Fl_\gamma, \mathbb{Z}) \quad = \quad \mathbb{Z}[x_1, x_2, \alpha]/(x_1^2 + x_2^2 - x_1x_2,\ 2\,\alpha - x_1^3,\ \alpha^2),$$

and

$$\begin{aligned} H^*(Fl_\gamma, \mathbb{Z}[\tfrac{1}{2}]) \quad &= \quad \mathbb{Z}[\tfrac{1}{2}][x_1, x_2]/(e_i(x_1^2, x_2^2, (x_1 - x_2)^2))_{i=1,2,3} \\ &= \quad \mathbb{Z}[\tfrac{1}{2}][x_1, x_2]/(x_1^2 + x_2^2 - x_1x_2,\ x_1^6). \end{aligned}$$

3.2.5. Twisting. Now we allow γ to take values in $L \cong M^{\otimes 3}$ for some line bundle M on X, so the corresponding bilinear form has values in $M^{\otimes 2}$. As described in [**Fu2**], this situation reduces to the case where L is trivial. Let $\widetilde{V} = V \otimes M^*$, so $\gamma : \bigwedge^3 V \to L$ determines a form $\widetilde{\gamma} : \bigwedge^3 \widetilde{V} \to k_X$. If $V = L_1 \oplus \cdots \oplus L_7$ is a γ-isotropic splitting as in Lemma 3.2.2, we have $\widetilde{V} = \widetilde{L}_1 \oplus \cdots \oplus \widetilde{L}_7$, where $\widetilde{L}_i = L_i \otimes M^*$. Thus

$$c(\widetilde{V}) = (1 - \widetilde{y}_1^2)(1 - \widetilde{y}_2^2)(1 - \widetilde{y}_3^2),$$

where $v = c_1(M)$, $\widetilde{y}_i = y_i - v$, so $\widetilde{y}_3 = \widetilde{y}_1 - \widetilde{y}_2 = y_1 - y_2$. Note that $y_1 - y_2 = y_3 - v$, since using γ and β there is a canonical isomorphism $L_2 \otimes L_3 \cong L_1 \otimes M$.

A rank 2 subbundle $E \subset V$ is γ-isotropic if and only if $\widetilde{E} = E \otimes M^* \subset \widetilde{V}$ is $\widetilde{\gamma}$-isotropic (a map is zero iff it is zero after twisting by a line bundle), so we have an isomorphism $\mathbf{Fl}_\gamma(V) \cong \mathbf{Fl}_{\widetilde{\gamma}}(\widetilde{V})$, and the tautological subbundles are related by $\widetilde{S}_i = S_i \otimes M^*$. Therefore $\widetilde{x}_i = -c_1(\widetilde{S}_i/\widetilde{S}_{i-1}) = x_i + v$. The presentation for $H^*\mathbf{Fl}_\gamma(V)$ is obtained from Proposition 3.2.6 by replacing y_i with $y_i - v$ and x_i with $x_i + v$.

CHAPTER 4

Giambelli formulas for Schubert loci

4.1. Divided difference operators and Chern class formulas

For now, assume γ takes values in the trivial bundle. Given $V \to X$ with a (complete) γ-isotropic flag of subbundles $E_\bullet$, Schubert loci $\mathbf{\Omega}_w \subset \mathbf{Fl}_\gamma(V)$ are defined as in §3.1.2, by the same conditions as when X is a point. Namely, set

$$\mathbf{\Omega}_w = \{x \in \mathbf{Fl}_\gamma \,|\, \mathrm{rk}(E_p(x) \to Q_q(x)) \leq r_w(q,p) \text{ for } 1 \leq q \leq 2,\ 1 \leq p \leq 7\},$$

where $Q_\bullet$ is the tautological flag of quotient bundles on $\mathbf{Fl}_\gamma(V)$. (Recall $r_w(q,p) = \#(i \leq q \,|\, w_i \leq p)$.) As usual, there are two steps to producing formulas for these Schubert loci: first find a formula for the most degenerate locus (the case $w = w_0$), and then apply divided difference operators to obtain formulas for all $w \leq w_0$. Theorem 4.1.1 and Lemma 4.1.3 prove Theorem 1.1.4.

THEOREM 4.1.1. *Assume M is trivial, and let $F_1 \subset F_2 \subset F_3 \subset V$ be a γ-isotropic flag. Then $[\mathbf{\Omega}_{w_0}] \in H^*\mathbf{Fl}_\gamma(V)$ is given by*

$$\begin{aligned}[\mathbf{\Omega}_{w_0}] &= \frac{1}{2}(x_1^3 - c_1(F_3)\,x_1^2 + c_2(F_3)\,x_1 - c_3(F_3)) \\ &\quad \times(x_1^2 + c_1(F_1)\,x_1 + c_2(F_3) - c_1(F_1)^2)(x_2 - x_1 - c_1(F_3/F_1)).\end{aligned}$$

Setting $y_1 = c_1(F_1)$ and $y_2 = c_1(F_2/F_1)$, we have $c(F_3) = (1+y_1)(1+y_2)(1+y_1-y_2)$, so this formula becomes $[\mathbf{\Omega}_{w_0}] = P_{w_0}(x;y)$, where

$$\begin{aligned}P_{w_0}(x;y) &= \frac{1}{2}(x_1^3 - 2\,x_1^2\,y_1 + x_1\,y_1^2 - x_1\,y_2^2 + x_1\,y_1\,y_2 - y_1^2\,y_2 + y_1\,y_2^2) \\ &\quad \times(x_1^2 + x_1\,y_1 + y_1\,y_2 - y_2^2)(x_2 - x_1 - y_2).\end{aligned}$$

PROOF. Let $p : \mathbf{Fl}_\gamma = \mathbb{P}(S_3/S_1) \to \mathcal{Q}$ be the projection. The locus where $S_1 = F_1$ is $p^{-1}\mathbb{P}(F_1)$, so its class is $p^*[\mathbb{P}(F_1)]$. On $\mathbb{P}(F_1) \subset \mathcal{Q}$, we have $S_1 = F_1$ and $S_3 = F_3$; thus on $p^{-1}\mathbb{P}(F_1)$, the locus where $S_2 = F_2$ is defined by the vanishing of the composed map $F_2/F_1 = F_2/S_1 \to S_3/S_1 \to S_3/S_2$. This class is given by $c_1((F_2/F_1)^* \otimes S_3/S_2) = x_2 - x_1 - c_1(F_2/F_1)$, so pushing forward by the inclusion $p^{-1}\mathbb{P}(F_1) \hookrightarrow \mathbf{Fl}_\gamma$, we have

$$[\mathbf{\Omega}_{w_0}] = p^*[\mathbb{P}(F_1)] \cdot (x_2 - x_1 - c_1(F_2/F_1)).$$

To determine $[\mathbb{P}(F_1)]$ in $H^*\mathcal{Q}$, we first find the class in $H^*\mathbb{P}(F_3)$ and then push forward. By [**Fu4**, Ex. 3.2.17], this is $x_1^2 + c_1(F_3/F_1)\,x_1 + c_2(F_3/F_1)$, and pushing forward is multiplication by $\alpha = [\mathbb{P}(F_3)]$. Using the relation given in §3.2.4, we have

$$\begin{aligned} [\mathbb{P}(F_1)] &= \alpha \cdot (x_1^2 + c_1(F_1)\, x_1 + c_2(F_3) - c_1(F_1)^2) \\ &= \frac{1}{2}(x_1^3 - c_1(F_3)\, x_1^2 + c_2(F_3)\, x_1 - c_3(F_3))(x_1^2 + c_1(F_1)\, x_1 + c_2(F_3) - c_1(F_1)^2). \end{aligned}$$

□

Recall (from §1.1.4) that the *divided difference operators* for G_2 are defined by

$$\begin{aligned} \partial_s(f) &= \frac{f(x_1, x_2) - f(x_2, x_1)}{x_1 - x_2}; \\ \partial_t(f) &= \frac{f(x_1, x_2) - f(x_1, x_1 - x_2)}{-x_1 + 2x_2} \end{aligned}$$

for the simple reflections, and by $\partial_w = \partial_{s_1} \circ \cdots \circ \partial_{s_\ell}$ if $w = s_1 \cdots s_\ell$ is a reduced expression. These operators may be constructed geometrically, using a correspondence as described in [**Fu1**]. Let $\mathcal{Q}(V)$ and $\mathcal{G}(V)$ be the quadric bundle and bundle of γ-isotropic 2-planes in V, respectively, and set $Z_s = \mathbf{Fl}_\gamma(V) \times_{\mathcal{G}(V)} \mathbf{Fl}_\gamma(V)$ and $Z_t = \mathbf{Fl}_\gamma(V) \times_{\mathcal{Q}(V)} \mathbf{Fl}_\gamma(V)$, with projections $p_i^s : Z_s \to \mathbf{Fl}_\gamma$ and $p_i^t : Z_t \to \mathbf{Fl}_\gamma$.

Lemma 4.1.2. *As maps $H^*\mathbf{Fl}_\gamma \to H^*\mathbf{Fl}_\gamma$,*

$$\begin{aligned} \partial_s &= (p_1^s)_* \circ (p_2^s)^* \text{ and} \\ \partial_t &= (p_1^t)_* \circ (p_2^t)^*. \end{aligned}$$

Proof. The proof is the same as in classical types. In the diagram

$$\begin{tikzcd} & Z_t \arrow[dl, "p_1^t"'] \arrow[dr, "p_2^t"] & \\ \mathbf{Fl}_\gamma \arrow[dr, "\pi"'] & & \mathbf{Fl}_\gamma \arrow[dl, "\pi"] \\ & \mathcal{Q}, & \end{tikzcd} \tag{4.1.1}$$

all maps are $\mathbb{P}^1$-bundles, so $(p_1^t)_* \circ (p_2^t)^* = \pi^* \circ \pi_*$. Since the universal bundle on $\mathbf{Fl}_\gamma = \mathbb{P}(S_3/S_1) \to \mathcal{Q}$ is S_2/S_1, π_* is determined by $\pi_*(x_2) = 1$. Setting $x_3 = x_1 - x_2$, the same algebra used in the type B_3 case shows $\pi^* \circ \pi_* = \partial_t$. The situation is similar for ∂_s. □

Lemma 4.1.3. *We have*

$$\partial_s[\mathbf{\Omega}_w] = \begin{cases} [\mathbf{\Omega}_{w\,s}] & \textit{if } \ell(w\,s) < \ell(w); \\ 0 & \textit{otherwise;} \end{cases}$$

and

$$\partial_t[\mathbf{\Omega}_w] \;=\; \begin{cases} [\mathbf{\Omega}_{w\,t}] & \text{if } \ell(w\,t) < \ell(w); \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Again, the proof is no different from the one for classical types. One immediately reduces to the case where X is a point, so $\mathbf{Fl}_\gamma(V) = Fl_\gamma$. Here one can use parametrizations of Schubert cells to see p_1 maps $p_2^{-1}\Omega_w$ into $\Omega_{w\,t}$ (or $\Omega_{w\,s}$), and this map is birational when $\ell(w\,t) < \ell(w)$ (respectively, $\ell(w\,s) < \ell(w)$). Alternatively, this is a general fact about G/P's (see Appendix A). □

By making the substitutions $x_i \mapsto x_i + v$ and $y_i \mapsto y_i - v$, we obtain formulas for the more general case, where γ has values in $M^{\otimes 3}$ for arbitrary M.

THEOREM 4.1.4. *Let $\gamma : \bigwedge^3 V \to M^{\otimes 3}$ be a nondegenerate form, with a γ-isotropic flag $F_\bullet \subset V$. Let $v = c_1(M)$. Let ∂_s be defined as above, and let ∂_t be given by*

$$\partial_t(f) \;=\; \frac{f(x_1,x_2) - f(x_1, x_1 - x_2 - v)}{-x_1 + 2x_2 + v}. \tag{4.1.2}$$

Then

$$[\mathbf{\Omega}_w] \;=\; P_w(x;y;v),$$

where $P_w = \partial_{w_0\,w^{-1}} P_{w_0}$, and

$$\begin{aligned} P_{w_0}(x;y;v) \;=\; & \frac{1}{2}(x_1^3 - 2\,x_1^2\,y_1 + x_1\,y_1^2 - x_1\,y_2^2 + x_1\,y_1\,y_2 - y_1^2\,y_2 + y_1\,y_2^2 \\ & +5\,x_1^2\,v - 7\,x_1\,y_1\,v + x_1\,y_2\,v + 2\,y_1^2\,v + y_1\,y_2\,v - 2\,y_2^2\,v \\ & +8\,x_1\,v^2 - 6\,y_1\,v^2 + 2\,y_2\,v^2 + 4\,v^3) \\ & \times(x_1^2 + x_1\,y_1 + y_1\,y_2 - y_2^2 + x_1\,v + y_2\,v)(x_2 - x_1 - y_2 + v). \end{aligned}$$

4.2. Variations

Any formula for the class of a degeneracy locus depends on a choice of representative modulo the ideal defining the cohomology ring; here we discuss some alternative formulas. In type A, the *Schubert polynomials* of Lascoux and Schützenberger are generally accepted as the best polynomial representatives for Schubert classes and degeneracy loci: they have many remarkable geometric and combinatorial (and aesthetic) properties. In other classical types, several choices have been proposed — see [**Bi-Ha, Kr-Ta, La-Pr, Fo-Ki, Fu2**] — but Fomin and Kirillov [**Fo-Ki**] gave examples showing that no choice can satisfy all the properties possessed by the type A polynomials. From this point of view, an investigation of alternative G_2 formulas could shed some light on the problem for classical types, by imposing some limitations on what one might hope to find for general Lie types.

PROPOSITION 4.2.1 (cf. [**Gr1**]). *Let*

$$\begin{aligned}\widetilde{P}_{w_0}(x;y) &= \frac{1}{54}(2\,x_1 - x_2 - y_1 + 2\,y_2)(2\,x_1 - x_2 - y_1 - y_2)(x_1 - 2\,x_2 + y_1 + y_2)\\ &\times(2\,x_1^3 - 3\,x_1^2x_2 - 3\,x_1x_2^2 + 2\,x_2^3 - 2\,y_1^3 + 3\,y_1^2y_2 + 3\,y_1y_2^2 - 2\,y_2^3).\end{aligned}$$

Then $[\mathbf{\Omega}_{w_0}] = \widetilde{P}_{w_0}(x;y)$ *in* $H^*\mathbf{Fl}_\gamma(V)$.

PROOF. Up to a change of variables, this is proved in [**Gr1**]. (To recover Graham's notation, set

$$(4.2.1)\qquad \begin{aligned} \xi_1 &= \tfrac{1}{3}(2x_1 - x_2), & \eta_1 &= -\tfrac{1}{3}(2y_1 - y_2),\\ \xi_2 &= \tfrac{1}{3}(-x_1 + 2x_2), & \eta_2 &= -\tfrac{1}{3}(-y_1 + 2y_2),\\ \xi_3 &= -\tfrac{1}{3}(x_1 + x_2), & \eta_3 &= \tfrac{1}{3}(y_1 + y_2),\end{aligned}$$

and replace ξ, η with x, y.) □

REMARK 4.2.2. In Graham's notation, $\widetilde{P}_{w_0} = -\frac{27}{2}(\xi_1 - \eta_2)(\xi_1 - \eta_3)(\xi_2 - \eta_3)(\xi_1\xi_2\xi_3 + \eta_1\eta_2\eta_3)$. This led him to suggest that $\frac{1}{2}(\xi_1\xi_2\xi_3 + \eta_1\eta_2\eta_3)$ might be an integral class. In fact, only 27 times this class is integral: Taking $[\Omega_w]^T = \widetilde{P}_w(x;t) = \partial_{w_0w^{-1}}\widetilde{P}_{w_0}(x;t)$, we compute

$$\frac{1}{2}(\xi_1\xi_2\xi_3 + \eta_1\eta_2\eta_3) = -\frac{1}{27}\left(3[\Omega_{tst}]^T + 3(t_1 + t_2)[\Omega_{st}]^T + (t_1 + t_2)(2t_1 - t_2)[\Omega_t]^T\right)$$

in $H_T^*(Fl_\gamma, \mathbb{Q})$; here the t's are related to the η's as in (4.2.1). (In fact, the two sides are equal as polynomials, not just as classes.) Since the equivariant Schubert classes $[\Omega_w]^T$ form a basis for $H_T^*(Fl_\gamma, \mathbb{Z})$ over $H_T^*(pt, \mathbb{Z}) = \mathbb{Z}[t_1, t_2]$, the right-hand side cannot be integral.

It is interesting to note that the integral class $-\frac{27}{2}(\xi_1\xi_2\xi_3 + \eta_1\eta_2\eta_3)$ is *positive* in the sense of [**Gr2**, Theorem 3.2]: the coefficients in its Schubert expansion are nonnegative combinations of monomials in the positive roots. It is therefore natural to ask whether this is the equivariant class of a T-invariant subvariety of Fl_γ. In fact, it is the class of a T-equivariant embedding of SL_3/B.[1]

REMARK 4.2.3. Graham's polynomial yields a simpler formula for the case where γ takes values in the trivial bundle, but $\det V = M$ is not necessarily trivial. (In this case, recall that $M^{\otimes 3}$ is trivial.) Making the substitutions $x_i \mapsto x_i + v$ and $y_i \mapsto y_i - v$, with $3\,v = 0$, we obtain

$$\begin{aligned}[\mathbf{\Omega}_{w_0}] &= \frac{1}{54}(2\,x_1 - x_2 - y_1 + 2\,y_2)(2\,x_1 - x_2 - y_1 - y_2)(x_1 - 2\,x_2 + y_1 + y_2)\\ &\times(2\,x_1^3 - 3\,x_1^2x_2 - 3\,x_1x_2^2 + 2\,x_2^3 - 2\,y_1^3 + 3\,y_1^2y_2 + 3\,y_1y_2^2 - 2\,y_2^3 + v^3).\end{aligned}$$

[1]This embedding projects to a $\mathbb{P}^2 \subset \mathcal{G}$. It is different from the embeddings of SL_3/B corresponding to the inlcusion of Lie algebras $\mathfrak{sl}_3 \subset \mathfrak{g}_2$, which project to $\mathbb{P}^2$'s in $\mathcal{Q}$.

There is a more transparent choice of polynomial representative for $[\Omega_{w_0}] \in H^*Fl_\gamma$ (i.e., the case where the base is a point): The class of a point in the 5-dimensional quadric $\mathcal{Q}$ is $\frac{1}{2}x_1^5$. Since Fl_γ is a $\mathbb{P}^1$ bundle over $\mathcal{Q}$, and x_2 is the Chern class of the universal quotient bundle, the class of a point in Fl_γ is $[\Omega_{w_0}] = \frac{1}{2}x_1^5 x_2$.

Starting from $\overline{P}_{w_0} = \frac{1}{2}x_1^5 x_2$, we can compute polynomials $\overline{P}_w$ for Schubert classes $[\Omega_w]$ using divided difference operators:

$$\overline{P}_{w_0} = \frac{1}{2}x_1^5 x_2$$

$$\overline{P}_{ststs} = \frac{1}{2}x_1^5$$

$$\overline{P}_{tstst} = \frac{1}{2}(x_1^3 + x_2 x_1^2 + x_2^2 x_1 + x_2^3)x_1 x_2$$

$$\overline{P}_{tsts} = \frac{1}{2}(4x_1^2 - 3x_1x_2 + 3x_2^2)x_1^2$$

$$\overline{P}_{stst} = \frac{1}{2}(x_1^4 + x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_2^4)$$

$$\overline{P}_{sts} = \frac{1}{2}(4x_1^2 - 3x_1x_2 + 3x_2^2)x_1$$

$$\overline{P}_{tst} = 2x_1^3 + \frac{1}{2}x_1^2 x_2 + \frac{1}{2}x_1 x_2^2 + 2x_2^3$$

$$\overline{P}_{ts} = 3x_1^2 - 2x_1x_2 + 2x_2^2$$

$$\overline{P}_{st} = 2x_1^2 - x_1x_2 + 2x_2^2$$

$$\overline{P}_{s} = x_1$$

$$\overline{P}_{t} = x_1 + x_2$$

$$\overline{P}_{id} = 1.$$

Remark 4.2.4. The classes in the left-hand column are pulled back from $\mathcal{Q}$, and it is easy to see that these polynomials are congruent to the formulas we know for these classes (that is, $x_1, x_1^2, \frac{1}{2}x_1^3, \frac{1}{2}x_1^4$, and $\frac{1}{2}x_1^5$) modulo the ideal. The appearance of complete homogeneous symmetric polynomials in the right-hand column can also be explained geometrically, using the embedding in the (type A) Grassmannian: $\mathcal{G} \subset Gr(2,7)$. Fix subspaces $F_6 = \langle f_1, \ldots, f_6 \rangle$ and $F_3' = \langle f_4, f_6, f_7 \rangle$. Let $Y \subset Gr(2,7)$ be the Schubert variety of 2-planes E which are contained in F_6, so $[Y] = x_1 x_2$ in H^*Gr. Let $Z \subset Gr(2,7)$ be the Schubert variety defined by $\dim(E \cap F_3') \geq 1$; this has class $h_3(x_1, x_2)$, where h is the complete homogeneous symmetric function. Using the parametrizations given in Appendix D.1, one can see that $Y \cap \mathcal{G} = X_{\overline{63}} = \Omega_{\overline{st}}$, which explains $\overline{P}_{st} \equiv x_1 x_2$. Moreover, the intersection of the cell $X^o_{\overline{63}}$ with Z is given by

$$\begin{pmatrix} a & -c^2 & 0 & c & d & 1 & 0 \\ c & -d & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \cap Z = \{a = -c^2 = d = 0\},$$

and Z does not intersect any other cell in $X_{\overline{63}}$, so $[Z]\cdot[Y]\cdot[\mathcal{G}] = 2\,[Z\cap Y\cap\mathcal{G}] = 2[pt]$ in $H^*Gr(2,7)$. It follows that $h_3(x_1,x_2)\cdot x_1x_2 = 2[pt]$ in $H^*\mathcal{G}$, which is expressed by the above formula for $\overline{P}_{tstst}$.

It is not possible to find a system of *positive* polynomials using divided difference operators. In this respect, the problem of "G_2 Schubert polynomials" is worse than the situation for types B and C: they cannot even satisfy two of Fomin-Kirillov's conditions [**Fo-Ki**].[2] Specifically, we have the following:

PROPOSITION 4.2.5. *Let $\{P_w \mid w\in W\}$ be a set of homogeneous polynomials, with $\deg P_w = \ell(w)$. Suppose*

$$\partial_s P_w = \begin{cases} P_{ws} & \text{when } \ell(w\,s) < \ell(w); \\ 0 & \text{when } \ell(w\,s) > \ell(w) \end{cases}$$

and

$$\partial_t P_w = \begin{cases} P_{wt} & \text{when } \ell(w\,t) < \ell(w); \\ 0 & \text{when } \ell(w\,t) > \ell(w). \end{cases}$$

Then for some w, P_w has both positive and negative coefficients.

PROOF. One just calculates, starting from $P_{id} = 1$, and finds that the positivity requirement leaves no choice in the polynomials up to degree 4:

$$\begin{array}{rcl} & P_{w_0} = ? & \\ P_{ststs} = ? & & \\ & & P_{tstst} = ? \\ P_{tsts} = ? & & \\ & & P_{stst} = \frac{1}{2}x_1^2x_2^2 \\ P_{sts} = \frac{1}{2}x_1^3 & & \\ & & P_{tst} = \frac{1}{2}(x_1^2x_2 + x_1x_2^2) \\ P_{ts} = x_1^2 & & \\ & & P_{st} = \frac{1}{2}(x_1^2 + x_1x_2 + x_2^2) \\ P_s = x_1 & & \\ & & P_t = x_1 + x_2 \\ & P_{id} = 1. & \end{array}$$

However, no degree 4 polynomial $P = P_{tsts}$ satisfies all the hypotheses. (Indeed, if $P = ax_1^4 + bx_1^3x_2 + \cdots + ex_2^4$, then $\partial_t P = 0$ implies $d = -2e$

[2]To be precise, the conditions we consider are [**Fo-Ki**, (3)] and a stronger version of [**Fo-Ki**, (1)].

and $b + c + d + e = 0$, hence $d = e = b = c = 0$. On the other hand, $\partial_s P = \frac{1}{2}(x_1^2 x_2 + x_1 x_2^2)$ requires $a = e$ and $b - d = \frac{1}{2}$, which is inconsistent with $b = c = d = e = 0$.) $\square$

In spite of this, one might look for polynomials which are positive in some other set of variables. One natural choice is to use x_1, x_2, and $x_3 = x_1 - x_2$; in fact, the polynomials given above (with $P_{w_0} = \frac{1}{2}x_1^5 x_2$) are positive in these variables.

CHAPTER 5

Degeneracy of morphisms

Let $\varphi : E \to F$ be a morphism of vector bundles on a smooth variety X, of ranks e and f respectively. When φ is sufficiently general, so the degeneracy locus

$$D_r(\varphi) = \{x \in X \mid \operatorname{rk} \varphi(x) \le r\} \subset X$$

has expected codimension (equal to $(e-r)(f-r)$), the Giambelli-Thom-Porteous formula gives the cohomology class $[D_r(\varphi)]$ in terms of the Chern classes of E and F.

In two cases of particular interest, there are also Chern class formulas for degeneracy loci where φ is not general. Taking $F = E^*$, one has the dual morphism $E^{**} = E \xrightarrow{\varphi^*} E^*$. Call φ **symmetric** if $\varphi^* = \varphi$, and **skew-symmetric** if $\varphi^* = -\varphi$. Such morphisms are not general in the above sense: the codimension of $D_r(\varphi)$ is at most $\binom{e-r+1}{2}$ (in the symmetric case) or $\binom{e-r}{2}$ (in the skew-symmetric case). Formulas for such loci were given by Harris-Tu [**Ha-Tu**] and Józefiak-Lascoux-Pragacz [**Jó-La-Pr**]. As described in [**Fe-Né-Ri**], these formulas can also be found by computing the equivariant classes of appropriate orbit closures in the $GL(E)$-representations $\operatorname{Sym}^2 E^*$ and $\bigwedge^2 E^*$, where E is a vector space.

In this chapter, we investigate the analogous problem for morphisms with *triality symmetry*, in the sense of Definition 1.1.5. In particular, we give two proofs of Theorem 1.1.6. The first approach is similar to that of Fehér and Rimányi, in that we study equivariant classes of orbit closures. Theorem 5.2.2 gives formulas for these classes, and implies Theorem 1.1.6.

We discuss a second approach to degeneracy loci for morphisms in §5.3. Given $\varphi : E \to F$, one constructs an auxiliary vector bundle $V = E \oplus F$ with appropriate bi- or trilinear forms. Then one considers morphisms whose graphs are isotropic subbundles of V, and deduces formulas for their degeneracy loci from formulas for Schubert loci in bundles. This approach was used by Fulton [**Fu3**] to find formulas (generalizing those of Harris-Tu) for symmetric and skew morphisms. The case of triality symmetry, however, is more subtle. Here $F = End(E) \oplus E^*$, and the auxiliary structure is an octonion algebra on $C = E \oplus F$, as in §2.4. Moreover, the graph of a triality-symmetric morphism need not be isotropic for the corresponding trilinear form; see Lemma 5.3.1.

5.1. Triality symmetry

Let E be a rank 2 vector bundle on a variety X. Recall (from §1.1.7) that a morphism $\varphi : E \to End(E) \oplus E^*$ is *triality-symmetric* if the corresponding section is a section of the subbundle

$$\left(\mathrm{Sym}^3 E^* \otimes \bigwedge^2 E\right) \oplus \bigwedge^2 E^* \subset Hom(E, End(E) \oplus E^*).$$

The terminology is motivated by Proposition B.4.1: locally, there is a $\mathbb{Z}/3\mathbb{Z}$-action on $\mathrm{Hom}(E, \mathrm{End}(E)) \oplus \bigwedge^2 E^* \subset \mathrm{Hom}(E, \mathrm{End}(E) \oplus E^*)$, fixing the subspace $\left(\mathrm{Sym}^3 E^* \otimes \bigwedge^2 E\right) \oplus \bigwedge^2 E^*$.

Suppose X is a point, so E is a vector space. Triality symmetry is described in terms of coordinates as follows. Choose a basis v_1, v_2 for E, and let v_1^*, v_2^* be the dual basis for E^*. Suppose $\varphi : E \to End(E) \oplus \bigwedge^2 E^*$ is given by $\varphi = \varphi_1 \oplus \varphi_2$, with

$$\begin{aligned} \varphi_1(v_1) &= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \\ \varphi_1(v_2) &= \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \end{aligned}$$

and $\varphi_2(v_1) = z\, v_2^*$, $\varphi_2(v_2) = -z\, v_1^*$.

Let $\{v_3, \ldots, v_8\}$ be a basis as in (2.4.1) for $\mathrm{End}(E) \oplus E^*$. In terms of these bases, φ is given by the matrix A_φ^t, where

$$A_\varphi = \begin{pmatrix} b_1 & a_1 & d_1 & c_1 & z & 0 \\ b_2 & a_2 & d_2 & c_2 & 0 & -z \end{pmatrix}.$$

Identify $\mathrm{Hom}(E, End(E)) = E^* \otimes E^* \otimes E$ with $E^* \otimes E^* \otimes E^*$ by mapping

$$\begin{aligned} v_i^* \otimes v_j^* \otimes v_1 &\mapsto v_{ij2}^*, \\ v_i^* \otimes v_j^* \otimes v_2 &\mapsto -v_{ij1}^*, \end{aligned}$$

where $v_{ijk}^* = v_i^* \otimes v_j^* \otimes v_k^*$ for $1 \le i, j, k \le 2$. Thus φ is triality-symmetric iff the corresponding coordinates of v_{ijk}^* are invariant under permutations of the indices. This means that the triality-symmetric maps φ are those whose matrix is of the form

$$A_\varphi = \begin{pmatrix} a & -d & d & c & z & 0 \\ b & a & -a & d & 0 & -z \end{pmatrix}. \tag{5.1.1}$$

(Here a is also the coordinate of v_{122}^*, b is the coordinate of v_{222}^*, $-c$ is the coordinate of v_{111}^*, and $-d$ is the coordinate of v_{112}^*.)

5.2. Orbits

In this section we compute the classes of orbit closures in the equivariant cohomology of a certain vector space; the connection with degeneracy loci is explained in Appendix C. See [**Fu5**] for basic facts about equivariant cohomology. (For fields other than $\mathbb{C}$, one should use the equivariant Chow groups defined in [**Ed-Gr2**].)

Let $G = G_2$ be the simple group of type G_2, and fix a maximal torus T. Choose simple roots α_1 and α_2, with α_2 long. Let B be the corresponding Borel subgroup, and let $P = P_{\widehat{2}}$ be the maximal parabolic omitting α_2. Let $\mathfrak{g}$, $\mathfrak{b}$, etc., denote the Lie algebras, so $\mathfrak{p} = \mathfrak{b} \oplus \mathfrak{g}_{-\alpha_1} \subset \mathfrak{g}$. Let $P = L \cdot P_u$ be the Levi decomposition, with P_u the unipotent radical and L a Levi subgroup; L is isomorphic to GL_2.

We will be interested in $\mathfrak{g}/\mathfrak{p}$ as a P-module and as an L-module. As an L-module, we have

$$\mathfrak{g}/\mathfrak{p} \cong \left(\operatorname{Sym}^3 E^* \otimes \textstyle\bigwedge^2 E\right) \oplus \textstyle\bigwedge^2 E^*,$$

where $E \cong \mathbb{C}^2$ is the standard representation of $L \cong GL_2$ (normalized to have weights $\alpha_1 + \alpha_2$ and $2\alpha_1 + \alpha_2$). As a P-module, $\mathfrak{g}/\mathfrak{p}$ does not split, but there is an exact sequence

$$0 \to \operatorname{Sym}^3 E^* \otimes \textstyle\bigwedge^2 E \to \mathfrak{g}/\mathfrak{p} \to \textstyle\bigwedge^2 E^* \to 0.$$

The T-weights of $\mathfrak{g}/\mathfrak{p}$ are

$$-\alpha_2,\ -\alpha_1 - \alpha_2,\ -2\alpha_1 - \alpha_2,\ -3\alpha_1 - \alpha_2,\ -3\alpha_1 - 2\alpha_2. \tag{5.2.1}$$

If we write $t_1 = -2\alpha_1 - \alpha_2$ and $t_2 = -\alpha_1 - \alpha_2$, the weights are

$$-t_1 + 2t_2,\ t_2,\ t_1,\ 2t_1 - t_2,\ t_1 + t_2. \tag{5.2.2}$$

We wish to compute the classes of P-orbits in $H_T^*(\mathfrak{g}/\mathfrak{p})$, so as a first step we give explicit descriptions of these orbits.

By the classification given in [**Jü-Rö**], there are finitely many P-orbits on $\mathfrak{g}/\mathfrak{p}$. In fact, there are five orbits. To describe them, let $W = \mathfrak{g}/\mathfrak{p}$ and $U = \operatorname{Sym}^3 E^* \otimes \bigwedge^2 E \subset W$. Let b, a, d, c be coordinates on U, with weights $-\alpha_2, -\alpha_1 - \alpha_2, -2\alpha_1 - \alpha_2, -3\alpha_1 - \alpha_2$, respectively. The five orbits are O_c, with $c = 0, 1, 2, 3, 5$ giving the codimension; their closures are nested and described by the following proposition:

PROPOSITION 5.2.1. *The P-orbit closures in $W = \mathfrak{g}/\mathfrak{p}$ are as follows:*

- $\overline{O_0} = W$.
- $\overline{O_1} = U$.
- $\overline{O_2}$ *is the discriminant locus in U, defined by the vanishing of the quartic polynomial* $a^2d^2 + 4a^3c + 4bd^3 - 27b^2c^2 + 18abcd$.
- $\overline{O_3}$ *is the (affine) cone over the twisted cubic curve in $\mathbb{P}^3 = \mathbb{P}U$, defined by the condition that the matrix*

$$\begin{pmatrix} a & -d & c \\ b & a & d \end{pmatrix}$$

have rank 1.
- $\overline{O_5} = O_5 = \{0\}$.

PROOF. The first claim is that $W \smallsetminus U = O_0$ is a single dense orbit. This follows from the classification of [**Bü-He**, Table 2].

It remains to verify the orbit decomposition of U. From the weights, we see that P_u acts trivially on U, so the effective action is by $P/P_u \cong GL_2$. Identify U with the space of homogeneous cubic polynomials in two variables: $U = \{-cx^3 - dx^2y + axy^2 + by^3\}$, with GL_2 acting so that the weights on a, b, c, d are as specified before the proposition. We see that there are four orbits in U: the polynomials with distinct roots, those with a double root, those with a triple root, and the zero polynomial. The given equations for the closures of these loci are well known; see e.g., [**La**, IV, Ex. 12(b)] for the discriminant and [**Fu-Pr**, §1.1] for the cubic curve. The proposition follows. $\square$

From the description in terms of cubic polynomials, it is easy to find representatives for orbits in U. Here we give representatives as weight vectors in $\mathfrak{g}/\mathfrak{p}$. Let $Y_\alpha \in \mathfrak{g}/\mathfrak{p}$ be a weight vector for α. We have

$$\begin{aligned}
O_0 &= P \cdot Y_{-3\alpha_1 - 2\alpha_2} = W \smallsetminus U;\\
O_1 &= P \cdot (Y_{-3\alpha_1 - \alpha_2} + Y_{-\alpha_2}) \cong P/P_u \cong GL_2;\\
O_2 &= P \cdot Y_{-\alpha_1 - \alpha_2};\\
O_3 &= P \cdot Y_{-\alpha_2};\\
O_5 &= \{0\}.
\end{aligned}$$

Using Proposition 5.2.1, it is a simple matter to compute the equivariant classes.

THEOREM 5.2.2. *In $H_T^*(W) = \mathbb{Z}[\alpha_1, \alpha_2] = \mathbb{Z}[t_1, t_2]$, we have*

$$\begin{aligned}
[\overline{O_0}] &= 1\\
[\overline{O_1}] &= -3\alpha_1 - 2\alpha_2\\
&= t_1 + t_2\\
[\overline{O_2}] &= 2(-3\alpha_1 - 2\alpha_2)^2\\
&= 2(t_1 + t_2)^2;\\
[\overline{O_3}] &= -3(\alpha_1 + \alpha_2)(2\alpha_1 + \alpha_2)(3\alpha_1 + 2\alpha_2)\\
&= 3t_1t_2(t_1 + t_2);\\
[\overline{O_5}] &= -\alpha_2(\alpha_1 + \alpha_2)(2\alpha_1 + \alpha_2)(3\alpha_1 + \alpha_2)(3\alpha_1 + 2\alpha_2)\\
&= t_1t_2(t_1 + t_2)(2t_1 - t_2)(-t_1 + 2t_2).
\end{aligned}$$

PROOF. The normal space to $U = \overline{O_1} \subset W$ has weight $-3\alpha_1 - 2\alpha_2$, so the formula for $[\overline{O_1}]$ is clear. Since the restriction $H_T^*(W) \to H_T^*(U)$ is an isomorphism, the Gysin pushforward $H_T^*(U) \to H_T^*(W)$ is multiplication by $[U]$. Therefore it suffices to compute the remaining classes in $H_T^*(U)$. The locus $\overline{O_2}$ is a hypersurface in U defined by an equation of weight $-6\alpha_1 - 4\alpha_2$, so its class in $H_T^*(W)$ is $(-6\alpha_1 - 4\alpha_2) \cdot [U]$. The class of $[\overline{O_3}]$ in $H_T^*(U)$ is found by the classical Giambelli (or Salmon–Roberts) formula. Finally, the class of the origin is the product of all the T-weights on W. $\square$

Remark 5.2.3. These classes cannot be computed by the "restriction equation" method of Fehér and Rimányi [**Fe-Ri2**], because the stabilizer of $O_1 = P/P_u$ is unipotent. This means the restriction map $H_P^*(W) \to H_P^*(O_1) \cong H_{P_u}^*(pt) = H^*(pt)$ is zero in positive degees, and all the restriction equations are of the form $0 = 0$. (The problem persists for the other orbits.)

The formulas of Theorem 1.1.6 may be deduced from those of Theorem 5.2.2 using the following lemma:

Lemma 5.2.4. *The orbit*

$$O_3 \subset \mathfrak{g}/\mathfrak{p} \subset \mathrm{Hom}(E, \mathrm{End}(E) \oplus E^*)$$

consists of the triality-symmetric morphisms of rank 1.

Proof. First note that any rank 1 map φ must correspond to an element $\varphi_1 \oplus \varphi_2 \in W = U \oplus \bigwedge^2 E^*$ with $\varphi_2 = 0$, i.e., φ lies in U. (If $\varphi_2 \neq 0$, then φ surjects onto E^*.)

Now the action of P on U is the same as that of its Levi subgroup GL_2. The inclusion $P \hookrightarrow P_{\hat{2}} \subset GL_8$ induces an inclusion of Levi subgroups $GL_2 \hookrightarrow GL_2 \times GL_6$, and the latter acts by conjugation on $\mathrm{Hom}(E, \mathrm{End}(E) \oplus E^*)$, so it preserves ranks of morphisms. Therefore it will suffice to check that a representative for O_2 has rank 2, and a representative from O_3 has rank 1. For these, we use the coordinate description given in §5.1. Under the identification of U with the space of cubic polynomials, the monomial xy^2 corresponds to the basis vector v_{122}^*. The orbit is O_2 (since xy^2 has two distinct zeroes), and the corresponding matrix A_φ has $b = c = d = 0$ and $a \neq 0$; it is easy to see this means φ has rank 2. Similarly, x^3 corresponds to v_{111}^*, and the corresponding A_φ has $a = b = d = 0$ and $c \neq 0$, so φ has rank 1. $\square$

5.3. Graphs

For any morphism $\varphi : E \to F$, let $E_\varphi \subset E \oplus F$ be its graph, i.e., the subbundle whose fiber over x is $E_\varphi(x) = \{(v, \varphi(v)) \,|\, v \in E(x)\}$. If $\varphi : E \to E^*$ is symmetric, then its graph is isotropic for the *canonical skew-symmetric form* on $E \oplus E^*$, defined by $(v_1 \oplus f_1, v_2 \oplus f_2) = f_1(v_2) - f_2(v_1)$. Thus one obtains a map to the Lagrangian bundle of isotropic flags in $E \oplus E^*$, and formulas for the degeneracy loci of φ are deduced from formulas for Schubert loci; see [**Fu3**] or [**Fu-Pr**].

In this section, we consider morphisms $\varphi : E \to End(E) \oplus E^*$. By Proposition 2.4.1, there is a canonical octonion algebra structure on $E \oplus End(E) \oplus E^*$. We give formulas for degeneracy loci of morphisms whose graphs are isotropic with respect to this structure. In general such morphisms are not triality-symmetric (nor vice-versa). For rank 1 maps, however, the two notions agree.

LEMMA 5.3.1. *Suppose X is a point, and $\varphi : E \to \mathrm{End}(E) \oplus E^*$ is a triality-symmetric map, with matrix A_φ^t as in* (5.1.1)*:*

$$A_\varphi = \begin{pmatrix} a & -d & d & c & z & 0 \\ b & a & -a & d & 0 & -z \end{pmatrix}.$$

Then the graph E_φ is contained in $V \subset C$, and is γ-isotropic if and only if

$$a^2 + bd = ac + d^2 = ad - bc = 0. \tag{5.3.1}$$

PROOF. This is a straightforward verification, using the basis $\{v_i\}$ as in §5.1. After a suitable change of coordinates (including a switch to opposite Schubert cells), the parametrization of the open Schubert cell given in §D.1 becomes

$$\widetilde{\Omega}^o = \begin{pmatrix} 1 & 0 & a & -d & d & c & z & -X \\ 0 & 1 & b & a & -a & d & -Z & -Y \end{pmatrix}, \tag{5.3.2}$$

where $X = -ac - d^2$, $Y = z + ad - bc$, and $Z = -a^2 - bd$. It is clear that the row span is always in $V \subset C$, since the fourth and fifth columns add to zero. (In the v-basis, V is defined by $v_4^* + v_5^* = 0$.) The condition that the row span be the graph E_φ means $X = Z = 0$ and $Y = z$, which are precisely the equations (5.3.1). □

COROLLARY 5.3.2. *Let $\varphi : E \to End(E) \oplus E^*$ be a morphism of rank at most* 1*, and such that the component $\varphi_2 : E \to E^*$ is zero. Then φ is triality-symmetric if and only if $E_\varphi \subset C$ is contained in $V \subset C$ and γ-isotropic. (This holds scheme-theoretically, i.e., the equations locally defining these two subsets of* $\mathrm{Hom}(E, \mathrm{End}(E))$ *are the same.)*

PROOF. This is a local statement, so we may assume X is a point and compute in coordinates. In this case, it follows from Lemma 5.3.1 by adding the equation $z = 0$. (The rank condition is forced by $\varphi_2 \equiv 0$.) □

Taking X to be a point, Corollary 5.3.2 says $\overline{O_3} = \widetilde{\Omega}^o \cap \Omega_{tst}$. It follows that the formulas for degeneracy of morphisms with γ-isotropic graphs are the same as those of Theorem 1.1.6.

THEOREM 5.3.3. *Let $\varphi : E \to End(E) \oplus E^*$ be a morphism, and suppose its graph E_φ is γ-isotropic in V. If X is Cohen-Macaulay and $D_r(\varphi)$ has expected codimension, then $[D_r(\varphi)] = P_r(c_1, c_2)$, where P_r is the polynomial of Theorem 1.1.6. Namely,*

$$\begin{aligned} P_2 &= 1, \\ P_1 &= 3\,c_2\,c_1 = 3x_1x_2(x_1 + x_2), \\ P_0 &= c_2\,c_1\,(9\,c_2 - 2\,c_1^2) = x_1x_2(x_1 + x_2)(2x_1 - x_2)(-x_1 + 2x_2). \end{aligned}$$

PROOF. Let $\varphi : E \to End(E) \oplus E^*$ have γ-isotropic graph E_φ. Suppose E has a rank 1 subbundle, so E_φ also does. Write $E_1 \subset E_2 = E$ and

$F_1 \subset F_2 = E_\varphi$, and extend these to complete γ-isotropic flags $E_\bullet$ and $F_\bullet$, as in §1.1.2. Write $Q_i = V/F_{7-i}$. For $w \in W(G_2)$, set

$$\Omega_w(\varphi) = \{x \in X \mid \mathrm{rk}(E_p(x) \to Q_q(x)) \leq r_w(q,p), 1 \leq p, q \leq 7\}.$$

Since $E_\varphi \cong E$, the Chern classes are the same. Let $-x_1, -x_2$ be Chern roots of E (so x_1, x_2 are Chern roots of E^*). Then as in §1.1.5, we have

$$[\Omega_w(\varphi)] = P_w(x_1, x_2; -x_1, -x_2) \tag{5.3.3}$$

in H^*X.

It remains to determine the w for which $D_r(\varphi) = \Omega_w(\varphi)$. We have

$$D_r(\varphi) = \{x \in X \mid \dim(E(x) \cap E_\varphi(x)) \geq 2 - r\}$$

(cf. §C.2.1), and it is easy to check that

$$\begin{aligned} D_2(\varphi) &= \Omega_{id}(\varphi) = \Omega_{1\,2}(\varphi) = X, \\ D_1(\varphi) &= \Omega_{tst}(\varphi) = \Omega_{3\,6}(\varphi), \\ D_0(\varphi) &= \Omega_{tstst}(\varphi) = \Omega_{6\,7}(\varphi). \end{aligned}$$

Indeed, $Q_5 = V/F_2$, so we have $\dim(E_2 \cap F_2) \geq 1$ iff $\mathrm{rk}(E_2 \to Q_5) \leq 1 = r_{3\,6}(2,5)$. The other two identities are clear.

Specializing the polynomials P_w given in §D.2 for these three w's, we obtain the desired formulas. $\square$

Remark 5.3.4. Up to sign, the twelve polynomials $P_w(x_1, x_2; -x_1, -x_2)$ are also the equivariant localizations $\sigma_w|_{w_0}$; see §D.3.

Remark 5.3.5. Over $\mathbb{C}$, Lemma 5.3.1 can also be deduced from Proposition C.3.4 as follows. The representation $\mathfrak{g}_2 \hookrightarrow \mathfrak{so}_8 \subset \mathfrak{gl}_8$ realizes $\mathfrak{n}^-$ as 8×8 matrices of the form

$$M = \left(\begin{array}{cc|cccc|c} & & & & & & \\ \hline a & b & & & & & \\ -d & a & & & & & \\ d & -a & & & & & \\ c & d & & & & & \\ \hline z & 0 & -d & -a & a & -b & \\ 0 & -z & -c & d & -d & -a & \end{array}\right), \tag{5.3.4}$$

with 0's in the blank blocks. (Here we are embedding $\mathfrak{so}_8$ in $\mathfrak{gl}_8$ using the basis $\{v_i\}$, so the bilinear form is given by (2.4.3).) One checks that the equations coming from $M^2 = 0$ are exactly those of (5.3.1).

CHAPTER 6

Characteristic two

The main results of this thesis continue to hold without restriction on the characteristic of the ground field. In this chapter, we explain the modifications necessary to include the case where $\mathrm{char}(k) = 2$. Throughout, k denotes an algebraically closed field of characteristic 2, except in Proposition 6.1.5, where k is arbitrary.

Most of the work to be done concerns the relationship between trilinear forms and octonions discussed in Chapter 2. Once the definitions are adjusted and corresponding facts about homogeneous spaces checked, the results of Chapters 3, 4, and 5 hold, with the same proofs.

6.1. Forms and octonions

All of the facts about composition algebras quoted in §2.1 hold in arbitrary characteristic; see [**Sp-Ve**, §§1–2].

Let V be a k-vector space of dimension 7. The definitions of compatible and nondegenerate forms given in Definition 1.1.1 become trivial when $\mathrm{char}(k) = 2$, so an alternate definition is needed. Using Lemma 2.2.4, we saw that there is a basis $f_1, \ldots, f_7$ for V such that a nondegenerate trilinear form is given by (2.2.8); we will take this as a definition in characteristic 2.

Definition 6.1.1. Fix a basis $f_1, \ldots, f_7$ for V. Let $\gamma_0 : \bigwedge^3 V \to k$ be defined by

$$\gamma_0 \;=\; f^*_{147} + f^*_{246} + f^*_{345} + f^*_{237} + f^*_{156}. \tag{6.1.1}$$

An alternating trilinear form γ is **nondegenerate** if it is equivalent to γ_0 (under the action of $GL(V)$).

Fix a basis $f_1, \ldots, f_7$ for V. Let γ_0 be as above, and define a symmetric bilinear form β_0 by

$$\begin{aligned} \beta_0(f_p, f_{8-q}) &= \delta_{pq}, \text{ for } p \neq 4 \text{ or } q \neq 4; \\ \beta_0(f_4, f_4) &= 0. \end{aligned} \tag{6.1.2}$$

We also need to specify the norm. In general, a quadratic norm N on a vector space W is called **nonsingular** if the corresponding quadric hypersurface $\mathcal{Q}(N) \subset \mathbb{P}(W)$ is nonsingular. (When $\mathrm{char}(k) \neq 2$, this is the same as requiring N to be nondegenerate, i.e., the associated bilinear form $\beta(u, v) = N(u+v) - N(u) - N(v)$ is nondegenerate; in characteristic 2, however, this bilinear form is alternating, so it is always degenerate if $\dim W$ is odd.) Using the basis $\{f_i\}$ for V, let N_0 be the nonsingular norm

$$N_0 = f_1^* f_7^* + f_2^* f_6^* + f_3^* f_5^* + (f_4^*)^2,$$

so the associated bilinear form is β_0. Note that β_0 is degenerate, with radical spanned by f_4. It induces a nondegenerate form $\overline{\beta}_0$ on $V/\langle f_4\rangle$.

DEFINITION 6.1.2. An alternating trilinear form γ and a nonsingular quadratic norm N are **compatible** if the pair (γ, N) is equivalent to (γ_0, N_0). A symmetric bilinear form β is compatible with γ if it is the bilinear form associated to a compatible norm.

A subspace $F \subseteq V$ is N-isotropic if the restriction $N|_F$ is identically zero; this condition replaces β-isotropicity in characteristic 2. The definition of γ-isotropicity is the same: when $\dim F \geq 2$, the subspace $F \subset V$ is γ-isotropic if the induced map $F \otimes F \to V^*$ is zero; a one-dimensional subspace is γ-isotropic if it is N-isotropic for a compatible norm N.

The analogue of Proposition 2.2.1 is the following:

PROPOSITION 6.1.3. *Given a compatible pair (γ, N) on V, there is an octonion algebra C with norm N' such that $V = e^\perp \subset C$, with $\gamma = \gamma'|_V$ and $N = N'|_V$, where γ' is the trilinear form induced by multiplication on C.*

PROOF. Choose a basis $\{f_i\}$ so that the pair is (γ_0, N_0). Let $E = \langle f_1, f_2\rangle$, and let $C = E \oplus \mathrm{End}(E) \oplus E^*$ be equipped with the canonical octonion structure, as in §2.4. Let $\{v_1, \ldots, v_8\}$ be the basis for C as in (2.4.1). Mapping

$$\begin{aligned}
f_1 &\mapsto v_1\\
f_2 &\mapsto v_2\\
f_3 &\mapsto v_3\\
f_4 &\mapsto v_4 - v_5\\
f_5 &\mapsto v_6\\
f_6 &\mapsto v_7\\
f_7 &\mapsto v_8
\end{aligned}$$

embeds V in C. It is straightforward to check the statements about restriction of forms, and that $V = e^\perp$. □

REMARK 6.1.4. This construction clearly works for any ground field k. Note that the subspace $E \subset V$ is γ-isotropic, and as a subspace of C, $uv = 0$ for all $u, v \in E$. Moreover, the bilinear form β' associated to N' on C is nondegenerate, and

$$\gamma'(u, v, w) = \beta'(uv, w) \tag{6.1.3}$$

for all $u, v, w \in C$. When $\mathrm{char}(k) = 2$, the basis element f_4 in V maps to e in C.

The above construction apparently depends on the choice of the γ-isotropic subspace $E \subset V$; we shall see that this is not the case. The following is another version of Proposition A.2.2, but here the ground field k is arbitrary.

PROPOSITION 6.1.5. *Let (γ, N) be a compatible pair on V, and let C be an octonion algebra as in Proposition 6.1.3. Let $G(\gamma, N) \subseteq GL(V)$ be the subgroup preserving γ and N. Then $G(\gamma, N) = \mathrm{Aut}(C)$. In particular, $G(\gamma, N)$ is simple of type G_2, and $GL(V)$ has a dense orbit in $\bigwedge^3 V^*$.*

PROOF. Proposition 6.1.3 shows that $\mathrm{Aut}(C) \subseteq G(\gamma, N)$. On the other hand, we claim that $G(\gamma, N)$ is semisimple of rank 2; since $\mathrm{Aut}(C)$ is also such a group, the proposition will follow. First, we may assume $(\gamma, N) = (\gamma_0, N_0)$, as before. The faithful $G(\gamma_0, N_0)$-representation $V/\langle f_4 \rangle$ is irreducible, since it is irreducible as a representation of the subgroup $\mathrm{Aut}(C)$ [**Sp-Ve**, Theorem 2.3.3]. It follows that $G(\gamma_0, N_0)$ is reductive [**Sp**, Ex. 2.4.15].

Let $T' \subset GL(V)$ be the maximal torus diagonal with respect to the basis $\{f_i\}$, so $T' \cong (k^*)^7$. The subgroup of T' preserving γ_0 is $T \times \mu_3$, where $T \cong (k^*)^2 \subset (k^*)^7$ is the two-dimensional torus

$$T = \{(z_1, z_2, z_1 z_2^{-1}, 1, z_1^{-1} z_2, z_2^{-1}, z_1^{-1})\}.$$

Requiring that the subgroup also preserve N leaves only T. Note that T is a regular torus in $GL(V)$, i.e., the centralizer $C_{GL(V)}(T)$ is equal to T'. Therefore T is a maximal torus in $G(\gamma, N)$, so $G(\gamma, N)$ has semisimple rank 2. To see $G(\gamma, N)$ is semisimple, note that it has trivial center. Indeed, the center must be a subgroup of T, but one checks that T lies in $\mathrm{Aut}(C)$, so it is a maximal torus of the simple group $\mathrm{Aut}(C)$.

The last statement follows from a dimension count. □

REMARK 6.1.6. A classification of trilinear forms on a seven-dimensional vector space over an arbitrary field is given in [**Co-He**]; the form γ_0 is equivalent to the one discussed in (3.9) of that article. The proof given there fails when $\mathrm{char}(k) = 2$, however, since the bilinear form associated to a norm is degenerate.

It follows that the construction of the composition algebra C is canonical:

COROLLARY 6.1.7. *Let (γ, N) be a compatible pair on V. Suppose $\{f_i\}$ and $\{f_i'\}$ are two bases for V for which (γ, N) has the form (γ_0, N_0). Let $E = \langle f_1, f_2 \rangle$ and $E' = \langle f_1', f_2' \rangle$, and let $V \hookrightarrow C$ and $V \hookrightarrow C'$ be the corresponding composition algebras. Then there is a unique isomorphism $C \xrightarrow{\sim} C'$ making the following diagram commute:*

$$\begin{tikzcd} C \arrow[rr, "\sim"] & & C' \\ & V \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}.$$

The analogue of Lemma 2.2.3 holds:

LEMMA 6.1.8. *Let (γ, N) be compatible on V, defining a composition algebra C. Then a subspace $L \subseteq V$ is γ-isotropic iff $uv = 0$ for all $u, v \in L$. In particular, any γ-isotropic subspace is also N-isotropic.*

PROOF. This is clear for one-dimensional subspaces, since by definition a one-dimensional space $\langle u \rangle$ is γ-isotropic iff $N(u) = 0$, and $N(u)e = u\overline{u} = -u^2$ for $u \in V$. So assume $\dim L > 1$. Let β be the bilinear form associated to N, and let γ' and β' be the extensions to C.

If L is C-isotropic, then for all $u, v \in L$, $\gamma'(u, v, \cdot) = \beta'(uv, \cdot) \equiv 0$ on C, so $\gamma(u, v, \cdot) \equiv 0$ on V, i.e., L is γ-isotropic.

Conversely, suppose L is γ-isotropic, and let $u, v \in L$. Then $\gamma(u, v, w) = \beta(uv, w) = 0$ for all $w \in V$, so $uv \in V^\perp = \langle e \rangle$. For contradiction, suppose $uv \neq 0$. Replacing u and v with scalar multiples, we may assume $uv = e$ and $N(u) = N(v) = 1$. But then $v = u^{-1} = \overline{u} = -u$, since $u \in V = e^\perp$. This contradicts the assumption $\dim L > 1$, so it follows that $uv = 0$.

The last statement comes from the fact that the norm is determined by $N(u)e = u\overline{u} = -u^2 (= u^2)$ for $u \in V$. $\square$

6.2. Forms and homogeneous spaces

Our next goal is to establish the analogue of Proposition 1.1.2. At the same time, we will show that Proposition A.4.1 holds in characteristic 2.

Let $\mathcal{Q} \subset \mathbb{P}(V)$ be the quadric defined by N, let $\mathcal{G} \subset Gr(2, V)$ be the subvariety parametrizing 2-dimensional γ-isotropic subspaces, and let $Fl_\gamma \subset \mathcal{Q} \times \mathcal{G}$ be the incidence variety, as before. Observe that Proposition A.4.1(a) holds; its proof is independent of characteristic.

The analogue of Proposition 1.1.2 is the following:

PROPOSITION 6.2.1. *Let (γ, N) be compatible on V, defining a composition algebra C. Let $u \in V$ have norm 0. Then the subspace*

$$E_u = \{v \in V \mid uv = 0\}$$

is three-dimensional and N-isotropic. Moreover, every two-dimensional C-isotropic subspace of E_u contains u.

This follows from the analogues of Lemmas 2.2.8 and 2.2.9, whose proofs are the same as before:

LEMMA 6.2.2. *If $u \in V$ is a nonzero N-isotropic vector, then*

$$E_u = \{v \in V \mid uv = 0\}$$

is a three-dimensional N-isotropic subspace.

LEMMA 6.2.3. *Let $u \in V$ be a nonzero N-isotropic vector, and let $v, w \in E_u$ be such that $\{u, v, w\}$ is a basis. Then $vw = \lambda u$ for some nonzero $\lambda \in k$.*

Using Proposition 6.2.1, we have a rank 3 bundle S_3 on $\mathcal{Q}$, and Fl_γ is identified with the $\mathbb{P}^1$-bundle $\mathbb{P}(S_3/S_1) \to \mathcal{Q}$ as before. The proof of the rest of Proposition A.4.1 therefore goes through. It follows that $G = \mathrm{Aut}(C)$ acts transitively on the set of 2-dimensional γ-isotropic subspaces of V.

APPENDIX A

Lie theory

In this appendix, we recall general facts about representation theory and homogeneous spaces for linear algebraic groups, and apply them show the above description of the G_2 flag variety agrees with the Lie-theoretic one. We do not know suitable references for Propositions A.2.2, A.2.6, and A.4.1, so we give proofs. All the remaining facts are standard, and can be found in e.g. [**Fu-Ha**], [**Hu1**], [**Hu2**], [**Sp**], [**De**].

Most of these facts hold in arbitrary characteristic, so unless indicated otherwise, k is any algebraically closed field.

A.1. General facts

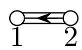
Let G be a simple linear algebraic group, fix a maximal torus and Borel subgroup $T \subset B \subset G$, and let $W = N(T)/T$ be the Weyl group. Let R, R^+, R^-, and Δ be the corresponding roots, positive roots, negative roots, and simple roots, respectively. For $\alpha \in \Delta$, let $s_\alpha \in W$ be the corresponding simple reflection, and let $\dot{s}_\alpha \in N(T)$ be a choice of lift. For a subset $S \subset \Delta$, let P_S be the parabolic subgroup generated by B and $\{\dot{s}_\alpha \,|\, \alpha \in S\}$. (Such parabolic subgroups are called *standard*.) For $P = P_S$, write $R^+(P) = R^+(S)$ for the set of positive roots which lie outside the span of the simple roots in S, and similarly for $R^-(P)$. Also write $\widehat{\imath} = \Delta \smallsetminus \{s_i\}$, so $P_{\widehat{\imath}}$ is the maximal parabolic in which the ith simple root is omitted. (For example, $SL_5/P_{\widehat{2}} \cong Gr(2,5)$.)

If $P = P_S$ is (standard) parabolic, its *unipotent radical* $P_u \subset P$ is the maximal normal unipotent subgroup. A parabolic group admits a decomposition $P = L \cdot P_u$ with $L \subset P$ a reductive subgroup, called a *Levi subgroup*; L can be chosen (uniquely) to contain T. Write $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$, $\mathfrak{p}$, $\mathfrak{p}_u$, $\mathfrak{l}$ for the corresponding Lie algebras, and write $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$ for the root space decomposition. Thus $\mathfrak{p}_u = \bigoplus_{\alpha\in R^+(P)} \mathfrak{g}_\alpha$, and $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{p}_u$. (We sometimes use $\mathfrak{n}$ for $\mathfrak{p}_u$.) The subgroup $P^- = \dot{w}_0 P \dot{w}_0$ is the *opposite* parabolic subgroup; its Lie algebra is $\mathfrak{p}^- = \mathfrak{l} \oplus \mathfrak{p}_u^-$, where $\mathfrak{p}_u^- = \bigoplus_{\alpha\in R^+(P)} \mathfrak{g}_{-\alpha}$.

The irreducible representations of G are indexed by dominant weights; write V_λ for the representation corresponding to the dominant weight λ. In characteristic 0, if $p_\lambda \in \mathbb{P}(V_\lambda)$ is the point corresponding to a highest weight vector, then $G \cdot p_\lambda$ is the unique closed orbit, and is identified with $G/P_{S(\lambda)}$, where $S(\lambda)$ is the set of simple roots orthogonal to λ with respect to a W-invariant inner product. In positive characteristic, $G/P_{S(\lambda)}$ can still be embedded in $\mathbb{P}(V)$ for some representation with highest weight λ, but V need not be irreducible. (See [**Hu2**, §31] for these facts about representations in arbitrary characteristic.)

A.2. Representation theory of G_2

The Dynkin diagram of type G_2 is



with corresponding simple roots α_1 and α_2. The full root system can be drawn as in Figure 1.

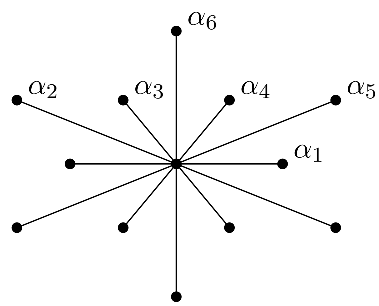


FIGURE 1. The root system of type G_2

The lattice of abstract weights is the same as the root lattice (cf. [**Hu2**, §A.9]); it follows that up to isomorphism, there is only one simple group of type G_2 (over the algebraically closed field k). From now on, let G denote this group, and fix $T \subset B \subset G$ corresponding to the root data. By Proposition 2.1.2, $G \cong \mathrm{Aut}(C)$, where C is the unique octonion algebra over k. Let $V = e^{\perp} \subseteq C$ be the imaginary subspace.

The dominant Weyl chamber for this choice of positive roots is the cone spanned by α_4 and α_6; denote these fundamental weights by ω_1 and ω_2, respectively. One checks that V has highest weight ω_1, and is irreducible for $\mathrm{char}(k) \neq 2$, so $V = V_{\omega_1}$ is the minimal irreducible representation, called the **standard** representation of G.[1] The adjoint representation $\mathfrak{g}$ has highest weight ω_2. (This is irreducible if $\mathrm{char}(k) = 0$, but not if $\mathrm{char}(k) = 3$.) Over any field, one has $\mathfrak{g} \subseteq \bigwedge^2 V$.

Let γ be the alternating trilinear form on $V \subset C$ induced by the multiplication, let $\{f_1, \ldots, f_7\}$ be the standard γ-isotropic basis (2.2.6), and let $E = \langle f_1, f_2 \rangle \subset V$. As we will see (Proposition A.4.1), the stabilizer of E in G is a maximal parabolic $P_{\widehat{2}}$, so we have $GL(E) \subset P_{\widehat{2}} \subset G$.

[1] If $\mathrm{char}(k) = 2$, the representation $V = e^{\perp} \subset C$ contains an invariant subspace spanned by e. In this case, the irreducible representation $V_{\omega_1} = V/(k \cdot e)$ is 6-dimensional [**Sp-Ve**, §2.3].

PROPOSITION A.2.1. *The subgroup $GL(E) \subset P_{\widehat{2}}$ is the Levi subgroup containing T.*

PROOF. From the root data, we see that the Levi factor of $\mathfrak{p}_2$ is $\mathfrak{gl}_2$. Indeed,

$$\mathfrak{p}_2 = (\mathfrak{t}_1 \oplus \mathfrak{t}_2 \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{-\alpha_1}) \oplus (\mathfrak{p}_2)_u,$$

with $[\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}] = \mathfrak{t}_1$. Since $GL(E)$ is a reductive subgroup with this Lie algebra, containing T, the proposition follows. $\square$

From the description of G as the automorphisms of C, it is clear that G preserves the alternating trilinear form γ. In fact, the converse is almost true:

PROPOSITION A.2.2. *Choose a basis $\{f_1, \dots, f_7\}$ for V, and let $\gamma \in \bigwedge^3 V^*$ be given by*

$$\gamma = f^*_{147} + f^*_{246} + f^*_{345} - f^*_{156} - f^*_{237},$$

as in (2.2.8)*. Let $G(\gamma) \subset GL(V)$ be the stabilizer of γ under the natural action, and let $SG(\gamma) = G(\gamma) \cap SL(V)$. Then $SG(\gamma)$ is simple of type G_2, and $G(\gamma) = \mu_3 \times SG(\gamma)$. Moreover, the orbit $GL(V) \cdot \gamma$ is open in $\bigwedge^3 V^*$.*

For $k = \mathbb{C}$, this is well known; see [**Br**, §2] or [**Fu-Ha**, §22]. For arbitrary fields, compare [**As**, (3.4)] and [**Co-He**, (2.1)]. We give a proof for $\mathrm{char}(k) \neq 2$.

PROOF. Fix an isomorphism $\iota : \bigwedge^7 V^* \cong k$, and let β_γ be as in (1.1.2). Since γ and β_γ are compatible, they define a composition algebra C (Proposition 2.2.1). Let $G = \mathrm{Aut}(C)$; this is simple of type G_2, and it is a subgroup of $SL(V)$ (Proposition 2.1.2).

Since γ may be recovered from the composition algebra structure as in (2.2.1), we have $G \subseteq SG(\gamma)$. On the other hand, $SG(\gamma)$ preserves β_γ, since it preserves ι. Therefore it preserves the algebra structure on C, so $SG(\gamma) = G$.

The group of scalars $\mathbb{G}_m \subset GL(V)$ acts on γ by $\lambda \cdot \gamma = \lambda^{-3}\gamma$, so $\mathbb{G}_m \cap G(\gamma) = \mu_3$. It follows that $G(\gamma) = \mu_3 \times SG(\gamma)$, so $\dim G(\gamma) = \dim SG(\gamma) = 14$.

Finally, the orbit $GL(V) \cdot \gamma \subset \bigwedge^3 V^*$ is isomorphic to the homogeneous space $GL(V)/G(\gamma)$. Since $\dim GL(V)/G(\gamma) = 49 - 14 = 35 = \dim \bigwedge^3 V^*$, the orbit is dense (and hence open). $\square$

The proof of this proposition also shows the following:

COROLLARY A.2.3. *Let V, γ, and $SG(\gamma)$ be as in Proposition A.2.2, and assume* $\mathrm{char}(k) \neq 2$*. Then $SG(\gamma)$ acts irreducibly on V.*

PROOF. We have seen that $SG(\gamma) = \mathrm{Aut}(C)$ for a composition algebra such that $V = e^\perp$, and this is an irreducible representation. $\square$

Remark A.2.4. Consider the map $\varphi : GL(V) \to \bigwedge^3 V^*$, $\varphi(g) = g \cdot \gamma$, and let $d\varphi : \mathfrak{gl}(V) \to \bigwedge^3 V^*$ be the differential. Let $\mathfrak{g}(\gamma) = \mathrm{Lie}(G(\gamma))$ and $\mathfrak{sg}(\gamma) = \mathrm{Lie}(SG(\gamma))$, so $\mathfrak{g}(\gamma) \subseteq \ker d\varphi$, and $\mathfrak{sg}(\gamma) \subseteq (\mathfrak{sl}(V) \cap \ker(d\varphi))$. If $\mathrm{char}(k) \neq 3$, a straightforward computation shows that $d\varphi$ is surjective, so $\dim\ker d\varphi = 14$. Proposition A.2.2 can be proved using this (cf. [**Fu-Ha**, p. 357]). If $\mathrm{char}(k) = 3$, $d\varphi$ is not surjective, but a similar computation shows $\dim\ker d\varphi = 15$, and $\dim(\mathfrak{sl}(V) \cap \ker(d\varphi)) = 14$.

Remark A.2.5. Our proof that $\bigwedge^3 V^*$ has an open orbit relies on computations with the specific form γ, and the apparently extraneous construction of a composition algebra. There is a more conceptual reason for the existence of open orbits, as follows. One shows that G_2 stabilizes some vector in $\bigwedge^3 V^*$, and that G_2 is not contained in any parabolic subgroup of $GL(V)$. By a general theorem of Röhrle, stabilizers of vectors in non-open orbits are always contained in parabolics.[2] See [**Ga3**, §9.12] for details and an application where the stabilizer is F_4.

Note that $w_0 \in W$ acts on the weight lattice by multiplication by -1. This implies that every irreducible representation of G is isomorphic to its dual. Using Schur's lemma, there is a unique (up to scalar) G-invariant bilinear form on each irreducible representation [**Hu2**, §31.6]. In particular, we have the following:

Proposition A.2.6. *Assume* $\mathrm{char}(k) \neq 2$. *Let* V *be a* 7*-dimensional vector space, with nondegenerate trilinear form* $\gamma : \bigwedge^3 V \to k$. *Then* γ *determines a compatible form* β *uniquely up to scaling by a cube root of unity.*

Proof. Let $G = SG(\gamma) \subset SL(V)$ be the subgroup preserving γ. Then G is simple of type G_2 and acts irreducibly on V, so the discussion above shows there is a G-invariant bilinear form β, unique up to scalar. Suppose β and $\lambda\beta$ are compatible with γ, for some $\lambda \in k^*$. If $\varphi^\dagger \in V$ is the inverse image of φ under the isomorphism $V \to V^*$ determined by β, and $\varphi^\ddagger$ is the inverse image under the isomorphism determined by $\lambda\beta$, then $\varphi^\ddagger = \lambda^{-1}\varphi^\dagger$. Thus we have

$$\begin{aligned} \gamma(u, v, \gamma(u, v, \cdot)^\ddagger) &= \frac{1}{2}\left(\lambda\beta(u,u)\,\lambda\beta(v,v) - (\lambda\beta(u,v))^2\right), \\ \lambda^{-1}\gamma(u, v, \gamma(u, v, \cdot)^\dagger) &= \frac{1}{2}\lambda^2\left(\beta(u,u)\beta(v,v) - \beta(u,v)^2\right), \end{aligned}$$

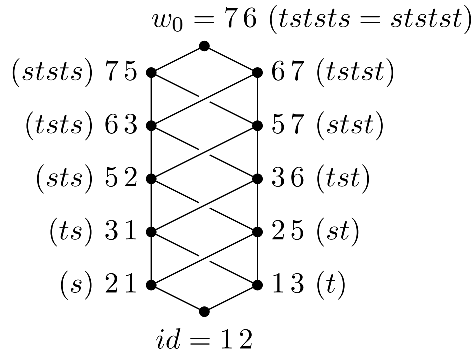
so $\lambda^3 = 1$. □

Remark A.2.7. In characteristic 0, the description of G_2 (or $\mathfrak{g}_2$) as the stabilizer of a generic alternating trilinear form is due to Engel, who also found an invariant symmetric bilinear form. For a history of some of the early constructions of G_2, see [**Ag**].

[2] If $\mathrm{char}\, k = 2$, this argument fails: in that case, G_2 has a faithful representation of degree 6, and GL_6 lies in the Levi factor of a maximal parabolic of $GL(V)$.

A.3. The Weyl group

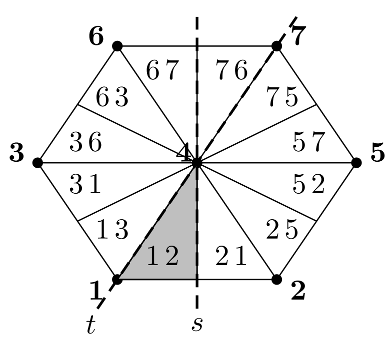
The Weyl group of type G_2 is the dihedral group with 12 elements. Let α_1 and α_2 be the simple roots, as in Figure 1, and let $s = s_{\alpha_1}$ and $t = s_{\alpha_2}$ be the corresponding simple reflections generating $W = W(G_2)$. Thus W has a presentation $\langle s, t \mid s^2 = t^2 = (st)^6 = 1\rangle$. With the exception of w_0, each element of $W(G_2)$ has a unique reduced expression. The Hasse diagram for Bruhat order is as follows:



The indexing $w = w_1\,w_2$, for $1 \le w_1, w_2 \le 7$, arises as follows. There is an embedding $W(G_2) \hookrightarrow W(A_6) = S_7$, given by $s \mapsto \tau_{12}\tau_{35}\tau_{67}$ and $t \mapsto \tau_{23}\tau_{56}$, where τ_{ij} is the permutation transposing i and j. (This also factors through $W(B_3)$.) Thus each w is identified with a permutation $w_1\,w_2 \cdots w_7$, and in fact, the full permutation is determined by $w_1\,w_2$.

This inclusion of Weyl groups corresponds to the inclusion $G_2 \hookrightarrow SL_7$ determined by the basis $\{f_1, \ldots, f_7\}$ for $V = V_{\omega_1}$ and the trilinear form γ of (2.2.8), together with the inclusion of tori $(z_1, z_2) \mapsto (z_1, z_2, z_1 z_2^{-1}, 1, z_1^{-1} z_2, z_2^{-1}, z_1^{-1})$. Thus a natural way to extend $w \in W$ to a full permutation is as follows. Given $w_1\,w_2$, let w_3 be the number such that $E_{f_{w_1}} = \langle f_{w_1}, f_{w_2}, f_{w_3}\rangle$ as in §3.1.1. Then define $w_4, \ldots, w_7$ by requiring $w_i + w_{8-i} = 8$. For example, $6\,3$ extends to $6\,3\,7\,4\,1\,5\,2$. Note that $(w \cdot w_0)_i = 8 - w_i$.

All this can be summarized in the following diagram:



A.4. Homogeneous spaces

We can now identify the homogeneous spaces for G_2. In this section, we allow k to have arbitrary characteristic. We take $G = \mathrm{Aut}(C)$ for an octonion algebra C, as above, and let β and γ be the corresponding compatible forms on the imaginary subspace $V \subset C$ (replacing β with the norm N if $\mathrm{char}(k) = 2$). From the root data, one sees $\dim G = 14$, $\dim B = 8$, $\dim P_{\hat{1}} = \dim P_{\hat{2}} = 9$, and $\dim T = 2$. Thus $\dim G/B = 6$ and $\dim G/P_{\hat{1}} = \dim G/P_{\hat{2}} = 5$.

Proposition A.4.1. *Let Fl_γ, $\mathcal{Q}$, and $\mathcal{G}$ be as in §3.1. Then*

(a) $\mathcal{Q} \cong G/P_{\hat{1}}$,
(b) $\mathcal{G} \cong G/P_{\hat{2}}$, *and*
(c) $Fl_\gamma \cong G/B$.

Proof. The homogeneous spaces $G/P_{\hat{1}}$ and $G/P_{\hat{2}}$ are the closed orbits in $\mathbb{P}(V)$ and $\mathbb{P}(\mathfrak{g})$, respectively. Since G preserves β (or N, in characteristic 2), $G/P_{\hat{1}}$ must be contained in the quadric hypersurface $\mathcal{Q} \subset \mathbb{P}(V)$, but $\dim G/P_{\hat{1}} = 5$, so it is all of $\mathcal{Q}$. This proves (a).

For (b), note that $G/P_{\hat{2}} \subset \mathbb{P}(\mathfrak{g}) \subset \mathbb{P}(\bigwedge^2 V)$, so $G/P_{\hat{2}} \subset Gr(2,7)$. Since G preserves γ, we must have $G/P_{\hat{2}} \subseteq \mathcal{G}$; thus it will suffice to show $\mathcal{G}$ is irreducible and 5-dimensional. For this, consider

$$Fl_\gamma = \{(p,\ell) \,|\, p \in \ell\} \subset \mathcal{Q} \times \mathcal{G},$$

and by Proposition 1.1.2 (which in turn used part (a)), the first projection identifies Fl_γ with the $\mathbb{P}^1$-bundle $\mathbb{P}(S_3/S_1) \to \mathcal{Q}$. Therefore Fl_γ is smooth and irreducible of dimension 6. On the other hand, the second projection is obviously a $\mathbb{P}^1$-bundle.

Finally, since Fl_γ is a 6-dimensional G-invariant subvariety of $G/P_{\hat{1}} \times G/P_{\hat{2}}$, it follows that $Fl_\gamma = G/B$, proving (c). □

Remark A.4.2. A similar description of $G/P_{\hat{2}}$, among others, can be found in [**La-Ma**].

We recall a few more facts for general G, P, B, T, W. The **length** of an element $w \in W$ is the least number $\ell = \ell(w)$ such that $w = s_1 \cdots s_\ell$ (with $s_j = s_{\alpha_j}$ for some $\alpha_j \in \Delta$); such a minimal expression for w is called a **reduced expression**. Write w_0 for the (unique) longest element of W. The **Bruhat order** on W is defined by setting $v \leq w$ if there are reduced expressions $v = s_{\beta_1} \cdots s_{\beta_{\ell(v)}}$ and $w = s_{\alpha_1} \cdots s_{\alpha_{\ell(w)}}$ such that the β's are among the α's.

For each $w \in W$, there is a **Schubert cell** $X_w^o = BwB/B$ in G/B, of dimension $\ell(w)$. The **Schubert varieties** X_w are the closures of cells, and $X_v \subseteq X_w$ iff $v \leq w$.

Let $B^- = w_0 B w_0$ be the opposite Borel, and let $N^- \subset B^-$ be its unipotent radical. Then the map $N^- \to w_0 \cdot X_{w_0}^o = B^- B/B$ is an isomorphism, identifying N^- with a neighborhood of eB in G/B (cf. [**Sp**, (8.3.6)]). The same holds for P a parabolic, with $N^- = P_u^- \subset P^-$ the unipotent radical of the opposite parabolic mapping isomorphically to the open set $B^-P/P \subset G/P$.

PROPOSITION A.4.3. *Let $i : G \hookrightarrow G'$ be an inclusion of semisimple algebraic groups, and let $B \subset G$ and $B' \subset G'$ be Borel subgroups with $i(B) \subset B'$. Also denote by i the induced inclusions of flag varieties $G/B \hookrightarrow G'/B'$ and Weyl groups $W \hookrightarrow W'$. Then for each $w \in W$, the Schubert cells are related by $BwB/B = (B'i(w)B'/B') \cap (G/B)$.*

More generally, let $P \subset G$ and $P' \subset G'$ be parabolic subgroups such that $P = P' \cap G$. Then the same conclusion holds for $G/P \hookrightarrow G'/P'$, that is, $BwP/P = (B'i(w)P'/P') \cap (G/P)$ for all $w \in W$.

PROOF. The inclusion $BwB/B \subseteq (B'i(w)B'/B') \cap (G/B)$ is clear. For the other direction, since the map i is B-equivariant, it is enough to note that the RHS is B-stable and contains only one T-fixed point (namely wB).

The proof for G/P is exactly the same, mutatis mutandis. □

A.5. The Borel map and divided differences

Let $M \subset \mathfrak{t}^*$ be the weight lattice. For general G/B, there is a **Borel map**

$$c : \mathrm{Sym}^* M \to H^*(G/B)$$

induced by the Chern class map $c_1 : M \to H^2(G/B)$, where $M \subset \mathfrak{t}^*$ is the weight lattice. More precisely, this map is defined as follows. Identify M with the character group of B, and associate to $\chi \in M$ the line bundle $L_\chi = G \times^B \mathbb{C}$. Then $c_1(\chi)$ is defined to be $c_1(L_\chi)$. (See [**BGG, De**].) In fact, c_1 is an isomorphism, and this induces an action of W in the evident way: for $w \in W$ and $x = c_1(\chi) \in H^2(G/B)$, define $w \cdot x = c_1(w \cdot \chi)$.

The Borel map becomes surjective after extending scalars to $\mathbb{Q}$, and sets up an isomorphism

$$H^*(G/B, \mathbb{Q}) \cong \mathrm{Sym}^* M_{\mathbb{Q}}/I,$$

where $I = (\mathrm{Sym}^* M_{\mathbb{Q}})_+^W$ is the ideal of positive-degree Weyl group invariants.

For a simple root α, define the **divided difference operator** ∂_α on $H^*(G/B)$ by

$$\partial_\alpha(f) = \frac{f - s_\alpha \cdot f}{\alpha}. \tag{A.5.1}$$

These act on Schubert classes as follows [**De**]:

$$\partial_\alpha[\Omega_w] \;=\; \begin{cases} [\Omega_{w\,s_\alpha}] & \text{when } \ell(w\,s_\alpha) < \ell(w); \\ 0 & \text{when } \ell(w\,s_\alpha) > \ell(w). \end{cases} \tag{A.5.2}$$

In particular, $[\Omega_{s_\alpha}]$ can be identified with the weight at the intersection of the hyperplanes orthogonal to α and the (affine) hyperplane bisecting α.

In the case of G_2 flags, we know $[\Omega_s] = x_1$ and $[\Omega_t] = x_1 + x_2$. Looking at the root diagram, then, we see $x_1 = \alpha_4$ and $x_2 = \alpha_3$. Therefore

$$\begin{aligned} \alpha_1 &= x_1 - x_2; \\ \alpha_2 &= -x_1 + 2x_2; \\ s \cdot x_1 &= x_2; \\ s \cdot x_2 &= x_1; \\ t \cdot x_1 &= x_1; \\ t \cdot x_2 &= x_1 - x_2. \end{aligned}$$

With these substitutions, the operators of (A.5.1) agree with those defined in §4.1 ((1.1.4) and (1.1.5)).

Remark A.5.1. The divided difference operators satisfy the braid relations defining W. (The algebra of divided differences is a representation of the *nil-Hecke algebra*.) Thus if $(\alpha_1, \ldots, \alpha_\ell)$ is a reduced word for w, one can define $\partial_w = \partial_{\alpha_1} \circ \cdots \circ \partial_{\alpha_\ell}$, and this is independent of the choice of word. It follows from (A.5.2) that for any class $\tau \in H^{2\ell}(G/B)$,

$$\tau = \sum_{\ell(w)=\ell} (\partial_w(\tau)) \cdot [\Omega_w].$$

In principle, this can be used to compute any given product in the cohomology of a flag variety; in practice, it is prohibitively inefficient. However, the G_2 flag variety is small enough to be treated by this brute-force method, which is how the computations in Remark 4.2.2 were carried out.

A.6. Equivariant vector bundles

Proposition A.6.1. *Let $H \subseteq G$ be a closed subgroup, and let U be an H-module. Then the functor*

$$U \mapsto G \times^H U,$$

with the evident action on morphisms, is an equivalence between the category of H-modules and the category of G-equivariant vector bundles on G/H. The functor taking an equivariant vector bundle to its fiber at $eH \in G/H$ is an inverse. □

Using this, we deduce some basic facts about vector bundles on $\mathcal{G} = G/P$, for $P = P_{\widehat{2}}$.

Corollary A.6.2. *The tautological rank 2 subbundle $S \subset V$ on $\mathcal{G}$ is the equivariant bundle*

$$S = G \times^P E,$$

where E is the 2-dimensional representation of P corresponding to the standard representation of the Levi subgroup $L \cong GL(E) \subset P$.

PROOF. By definition, the fiber of S over $eP = [E] \in \mathcal{G}$ is E, and P stabilizes E. We saw in Proposition A.2.1 that $GL(E)$ is a Levi subgroup of P. $\square$

COROLLARY A.6.3. *Let $S \subset V$ be the tautological rank 2 subbundle on $\mathcal{G}$, and let $i : \mathcal{G} \hookrightarrow Gr_{\beta'}(2, C)$ be the inclusion. Then the tangent bundles are described by the diagram below:*

$$\begin{tikzcd}
T\mathcal{G} \arrow[r, hook] \arrow[d, equal] & i^*TGr_{\beta'} \arrow[d, equal] \\
\left(\operatorname{Sym}^3 S^* \otimes \mathcal{O}_{\mathcal{G}}(-1)\right) \oplus \mathcal{O}_{\mathcal{G}}(1) \arrow[r, hook] & Hom(S, End(S)) \oplus \mathcal{O}_{\mathcal{G}}(1),
\end{tikzcd}$$

where $\mathcal{O}_{\mathcal{G}}(1) = \bigwedge^2 S^*$.

PROOF. Restrict to fibers over $[E] \in \mathcal{G}$, and use Proposition B.4.1. $\square$

APPENDIX B

Triality

If G is a simply-connected, split, semisimple group, then symmetries of the corresponding Dynkin diagram induce the outer automorphisms of G (see, e.g., [**Fu-Ha**, Prop. D.40] or [**KMRT**, §25.B]). The Dynkin diagram of type D_4 possesses a symmetry not shared by other Dynkin diagrams: the order 3 symmetry rotating the outer nodes. The resulting symmetry of $Spin_8$ is called *triality*.

Triality has a long and interesting history. In 1925, Cartan described the S_3 action on the root system of type D_4 and its relationship to a geometric *principe de trialité*, an incidence-preserving correspondence between points on a six-dimensional quadric and each of the two families of $\mathbb{P}^3$'s in the quadric [**Ca**]. (This geometric correspondence was also considered by Study in 1913.) Cartan also noted the connection with automorphisms of the octonions, and the fact that the fixed subalgebra for the corresponding action on $\mathfrak{so}_8$ is $\mathfrak{g}_2$. Since then, triality has been considered in several contexts by many mathematicians. For example, Freudenthal used triality to unify his construction of the exceptional Lie algebras; Springer and van der Blij clarified some of Cartan's remarks and gave formulations of triality valid over arbitrary fields; and triality plays a central role in the classification of forms of D_4 given in [**KMRT**].

In this chapter, we give a brief exposition of the manifestations of triality relevant to the symmetries of morphisms described in Chapter 5. We make no claims of originality: most of the ideas can be found in Cartan (without proofs). Other references for this material include [**vdB-Sp**], [**Sp-Ve**, §3], [**Ga1**], [**Fu-Ha**, §20.3], and [**KMRT**].

Throughout, k is an algebraically closed field of characteristic not 2.

B.1. D_4 flags

Let C be an 8-dimensional k-vector space, equipped with a nondegenerate symmetric bilinear form β'.[1] Here we describe the variety of isotropic flags in C, generally using the projective terminology: thus a *point* is a one-dimensional subspace of C, a *line* is a two-dimensional subspace, etc. Let $Q \subset \mathbb{P}(C) = \mathbb{P}^7$ be the 6-dimensional quadric defined by β', and let $Q_2 = Gr_{\beta'}(2, C)$ be the 9-dimensional Grassmannian of (projective) lines in

[1]Our notation is chosen to agree with that of earlier chapters, and in anticipation of §B.2, where we will equip C with the structure of an octonion algebra.

Q. Recall that there are two families of isotropic 4-dimensional subspaces of C (i.e., *spaces* in Q); let $Q^+ = Gr^+_{\beta'}(4,8)$ and $Q^- = Gr^-_{\beta'}(4,8)$ be these two families. (If one fixes a given space $\Sigma_0^+ \cong \mathbb{P}^3 \subset Q$, then Q^+ parametrizes spaces Σ^+ such that $\dim(\Sigma^+ \cap \Sigma_0)$ is odd, while Q^- parametrizes spaces Σ^- such that $\dim(\Sigma^- \cap \Sigma_0)$ is even.)

The D_4 flag variety is the incidence variety

$$Fl_{\beta'}(C) = \{(p, \ell, \Sigma^+, \Sigma^-) \,|\, p \in \ell \subset \Sigma^+ \cap \Sigma^-\} \subset Q \times Q_2 \times Q^+ \times Q^-.$$

This is a homogeneous projective variety of dimension 12. The parity condition implies that for any $(\Sigma^+, \Sigma^-) \in Q^+ \times Q^-$, the intersection $\Sigma^+ \cap \Sigma^-$ is either a point or a plane. For $(p, \ell, \Sigma^+, \Sigma^-) \in Fl_{\beta'}$, we have $\dim(\Sigma^+ \cap \Sigma^-) = 2$, since the intersection contains the line ℓ. Thus a D_4 flag also determines a plane, and there is a tautological flag of vector bundles

$$\begin{tikzcd}
 & & & S_4^+ \arrow[dr, hook] & & & & \\
S_1 \arrow[r, hook] & S_2 \arrow[r, hook] & S_3 \arrow[ur, hook] \arrow[dr, hook] & & S_5 \arrow[r, hook] & S_6 \arrow[r, hook] & S_7 \arrow[r, hook] & C \,, \\
 & & & S_4^- \arrow[ur, hook] & & & &
\end{tikzcd}$$

where $S_i = S_{8-i}^\perp$.

Remark B.1.1. From this construction, we see that there are two embeddings $\iota^+, \iota^- : Fl_{\beta'}(C) \hookrightarrow Fl(C)$ into the full flag variety, given by taking the middle flag to be S_4^+ or S_4^-, respectively. (This is also equivalent to the choice of Σ_0.) There is an involution on $Fl(C) \cong SL_8/B$ defined by $(E_\bullet) \mapsto (E_{8-\bullet}^\perp)$; the fixed locus has two connected components, corresponding to these two embeddings.

Let $pr_{134} : Fl_{\beta'} \to Q \times Q^+ \times Q^-$ and $pr_2 : Fl_{\beta'} \to Q_2$ be the projections, and let $Y \subset Q \times Q^+ \times Q^-$ be

$$Y = pr_{134}(Fl_{\beta'}) = \{(p, \Sigma^+, \Sigma^-) \,|\, \dim(\Sigma^+ \cap \Sigma^-) = 2 \text{ and } p \in \Sigma^+ \cap \Sigma^-\}.$$

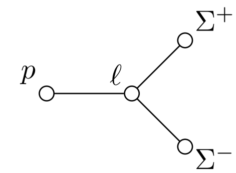
The projection pr_{134} realizes $Fl_{\beta'}$ as the $\mathbb{P}^1$ bundle $\mathbb{P}(S_3/S_1) \to Y$. The projection pr_2 realizes $Fl_{\beta'}$ as a $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$-bundle over Q_2: Indeed, we have

$$Fl_{\beta'} = \mathbb{P}(S_2) \times_{Q_2} \mathbf{Q}(S_6/S_2),$$

where S_2 is the tautological bundle on Q_2, $S_6 = S_2^\perp$, and $\mathbf{Q}(S_6/S_2)$ is the quadric bundle defined by the restriction of β'. (The fibers are quadric surfaces in $\mathbb{P}^3$, so isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.)

B.2. Geometric triality

The data parametrized by $Fl_{\beta'}(C)$ can be arranged suggestively on the D_4 Dynkin diagram as below.

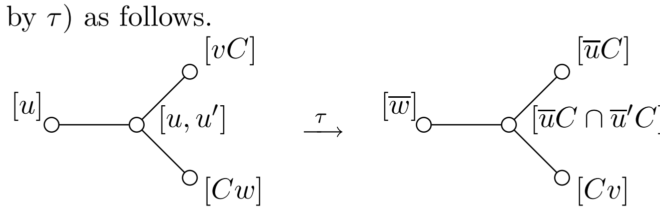


The *triality automorphism* $\tau : Fl_{\beta'}(C) \to Fl_{\beta'}(C)$ rotates this diagram; it also induces automorphisms of Q_2 and $Y \subset Q \times Q^+ \times Q^-$. To describe it, we use some facts about octonions.

Equip C with a split octonion algebra structure, with norm N corresponding to the bilinear form β'. (Recall that *split* means the norm is isotropic, i.e., there exist nonzero vectors of norm 0.) This can be done as follows. Choose a vector $e \in C$ of norm 1 to serve as the identity. Then choose a basis $f_1, \ldots, f_7$ for $V = e^\perp$ such that the restriction $\beta = \beta'|_V$ has the form (2.2.7), and define multiplication using the trilinear form γ of (2.2.8). Recall that *conjugation* is defined on C by $\overline{u} = \beta'(u, e)\, e - u$.

Lemma B.2.1 ([**vdB-Sp**, §2]). *Let $u \in C$ be a nonzero vector with $N(u) = 0$. Then uC and Cu are 4-dimensional isotropic subspaces in different families. Conversely, every maximal isotropic subspace occurs this way, and the vector defining it is unique up to scalar.*

Now we can define maps $Q_2 \to Q_2$, $Q \to Q^+$, $Q^+ \to Q^-$, and $Q^- \to Q$ (all denoted by τ) as follows.



The fact that incidence is preserved under this map is what Cartan calls the *principle of triality*; see also [**vdB-Sp**, Theorem 7].

It is easy to check that τ^3 is the identity, so τ gives an action of $\mathbb{Z}/3\mathbb{Z}$ on $Fl_{\beta'}(C)$. Together with the involution induced by $C \to C$, $u \mapsto \overline{u}$, this defines an action of the symmetric group S_3. We shall focus on the $\mathbb{Z}/3\mathbb{Z}$-action.

Let $V \subset C$ be the 7-dimensional subspace of imaginary octonions, i.e., $V = e^\perp$. Let γ be the restriction to V of the trilinear form defined by multiplication on C; recall that γ is an alternating form, and the G_2 flag variety is the variety of γ-isotropic flags in V (see §1.1.2). The following proposition is essentially well known, but the proof we give sheds some light on the tautological rank 3 bundle on $Fl_\gamma(V)$.

PROPOSITION B.2.2. *The fixed locus $Fl_{\beta'}(C)^\tau$ for the $\mathbb{Z}/3\mathbb{Z}$ action is the G_2 flag variety $Fl_\gamma(V)$. Similarly, the fixed locus for the action on $Q_2 = Gr_{\beta'}(2, C)$ is the G_2 Grassmannian $\mathcal{G}$.*

PROOF. First consider the action on Y. The point $([u], [vC], [Cw]) \in Y \subset Q \times Q^+ \times Q^-$ is fixed when $[u] = [\overline{w}] = [\overline{v}]$ (and in this case the component $[u]$ determines the components $[vC]$ and $[Cw]$). Now for $[u] \in [\overline{u}C]$, we must have $u \in V$. Indeed, suppose $u = \overline{u}x$ for some x in C. Then $u^2 = u(\overline{u}x) = (u\overline{u})x = N(u)x = 0$. By the minimal equation for u (see (2.1.1)), this implies $\beta(u, e) = 0$, i.e., $u \in V$. It follows that the fixed locus Y^τ is isomorphic to the 5-dimensional quadric $\mathcal{Q} \subset \mathbb{P}(V)$.

Next consider the action on Q_2. A line $[u, u']$ is fixed by τ when $[u, u'] = [\overline{u}C \cap \overline{u}'C]$. This is equivalent to the four conditions

$$\begin{aligned} u &\in \overline{u}C, \\ u' &\in \overline{u}'C, \\ u &\in \overline{u}'C, \\ u' &\in \overline{u}C. \end{aligned}$$

As discussed in the previous paragraph, the first two conditions are equivalent to $u, u' \in V \subset C$. The second two are equivalent to $uu' = 0$: If $u' = \overline{u}x$ for some x, then $uu' = u(\overline{u}x) = (u\overline{u})x = 0$. Conversely, the kernel of the map $x \mapsto ux$ is precisely $\overline{u}C$. (The condition $u \in \overline{u}'C$ is verified similarly.) Now by Lemma 2.2.3, a τ-fixed line is the same as a γ-isotropic line, i.e., a point of $\mathcal{G} = G_2/P_2$.

Finally, since incidence is preserved by τ, the proposition follows. □

REMARK B.2.3. The variety Y comes with three tautological subbundles of the trivial bundle C: S_1, S_4^+, and S_4^-, as well as the rank 3 bundle $S_3 = S_4^+ \cap S_4^-$. In the above proof, we saw that the restriction of S_1 to $Y^\tau \cong \mathcal{Q}$ is a subbundle of $V \subset C$. A symmetric argument shows that $S_3 \subset V$; this is precisely the rank 3 bundle described in Proposition 1.1.2.

B.3. Root systems

The geometric triality described above comes from an automorphism of the simply connected type-D_4 group $Spin_8$ which fixes the type G_2 subgroup $G = \mathrm{Aut}(C)$. (See [**Sp-Ve**, §3] for the details on this action and the "local triality" action on the corresponding Lie algebras.) This automorphism of $Spin_8$ is induced by an automorphism of the D_4 root system, which we now describe.[2]

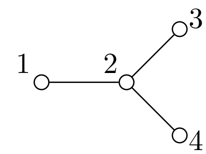
Let $T' \subset Spin_8$ be a maximal torus, with Lie algebra $\mathfrak{t}'$. The root system of type D_4 has 24 roots: $R = \{\pm t'_i \pm t'_j \,:\, 1 \le i < j \le 4\}$, where $\{t'_1, t'_2, t'_3, t'_4\}$

[2]We shall not use any spin representations explicitly, and the reader more comfortable with orthogonal groups may read $Spin_8$ as SO_8. However, it is important to remember that the $\mathbb{Z}/3\mathbb{Z}$ actions described in this chapter lift only to $Spin_8$ (or PSO_8), and not to SO_8.

is an orthonormal basis for $\mathfrak{t}'^*$ (with respect to a form invariant under the Weyl group). Choosing simple roots

$$\begin{aligned}\alpha'_1 &= t'_1 - t'_2\\ \alpha'_2 &= t'_2 - t'_3\\ \alpha'_3 &= t'_3 - t'_4\\ \alpha'_4 &= t'_3 + t'_4,\end{aligned}$$

the positive roots are $R^+ = \{t'_i \pm t'_j \,|\, i < j\}$. Label the D_4 Dynkin diagram as follows, with node i corresponding to the simple root α'_i:



The triality action on $\mathfrak{t}'^*$ is given by

$$\begin{aligned}&\tau : \alpha'_1 \mapsto \alpha'_4 \mapsto \alpha'_3 \mapsto \alpha'_1,\\ &\tau : \alpha'_2 \mapsto \alpha'_2.\end{aligned}$$

In the t'-basis, τ has matrix

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Note that $at'_1 + bt'_2 + ct'_3 + dt'_4$ is invariant under τ iff $d = 0$ and $a = b + c$, i.e., the invariant vectors are

$$b(t'_1 + t'_2) + c(t'_1 + t'_3).$$

The fixed positive roots are

$$\begin{aligned}t'_2 - t'_3 &= \alpha'_2,\\ t'_1 + t'_3 &= \alpha'_1 + \alpha'_2 + \alpha'_3 + \alpha'_4,\\ t'_1 + t'_2 &= \alpha'_1 + 2\,\alpha'_2 + \alpha'_3 + \alpha'_4.\end{aligned}$$

Now let $T \subset T'$ be a maximal torus of $G = G_2 = \mathrm{Aut}(C)$, so T is fixed by τ; let $\mathfrak{t}$ be its Lie algebra. Let $\alpha_1 = t_1 - t_2$ and $\alpha_2 = -t_1 + 2t_2$ be the simple roots, as in §A.2. The restriction map $\mathfrak{t}'^* \to \mathfrak{t}^*$ is given by $t'_1 \mapsto t_1$, $t'_2 \mapsto t_2$, $t'_3 \mapsto t_1 - t_2$, $t'_4 \mapsto 0$. In terms of the simple roots, this is

$$\begin{aligned}\alpha'_1, \alpha'_3, \alpha'_4 &\mapsto \alpha_1;\\ \alpha'_2 &\mapsto \alpha_2.\end{aligned}$$

To see this, it is easiest to use the basis $\{v_1, \ldots, v_8\}$ for C, as in (2.4.1). In this basis, T' acts with weights $\{t'_1, t'_2, t'_3, t'_4, -t'_4, -t'_3, -t'_2, -t'_1\}$, while T acts with weights $\{t_1, t_2, t_1 - t_2, 0, 0, t_2 - t_1, -t_2, -t_1\}$.

B.4. Tangent spaces

Let $i : \mathcal{G} \hookrightarrow Gr_{\beta'}(2, C)$ be the inclusion of the fixed locus for τ. Still using the basis $\{v_i\}$ for C, let $[E] \in Gr(2, C)$ be the point corresponding to $E = \langle v_1, v_2 \rangle$. We wish to compare the tangent spaces $T_{[E]}\mathcal{G}$ and $T_{[E]}Gr_{\beta'}$, both considered as subspaces of $T_{[E]}Gr(2, C) = \mathrm{Hom}(E, C/E)$.

Identify $\mathcal{G} = G/P$ and $Gr_{\beta'} = Spin_8/P'$, for $G = \mathrm{Aut}(C)$, and $P \subset G$, $P' \subset Spin_8$ maximal parabolic subgroups. Since $[E]$ corresponds to the point $eP \in G/P$ and $eP' \in Spin_8/P'$, we are considering

$$\mathfrak{g}/\mathfrak{p} \subset \mathfrak{so}_8/\mathfrak{p}' \subset \mathrm{Hom}(E, C/E).$$

Moreover, τ acts on $T_{[E]}Gr_{\beta'} = \mathfrak{so}_8/\mathfrak{p}'$, fixing the subspace $T_{[E]}\mathcal{G} = \mathfrak{g}/\mathfrak{p}$. Recall that $GL(E)$ is a Levi subgroup of $P \subset G$.

Proposition B.4.1. *As $GL(E)$-modules, the tangent spaces are described by the diagram below:*

$$\begin{tikzcd} T_{[E]}\mathcal{G} \arrow[r, hook] \arrow[d, equal] & T_{[E]}Gr_{\beta'} \arrow[d, equal] \\ \left(\mathrm{Sym}^3 E^* \otimes \bigwedge^2 E\right) \oplus \bigwedge^2 E^* \arrow[r, hook] & \mathrm{Hom}(E, \mathrm{End}(E)) \oplus \bigwedge^2 E^*. \end{tikzcd}$$

Proof. Representations of $GL(E) \cong GL_2$ are determined by their weights. The weights of T' acting on $T_{[E]}Gr_{\beta'} = \mathfrak{so}_8/\mathfrak{p}'$ are the 9 roots

$$\begin{array}{lll} -\alpha'_2, & -\alpha'_1 - \alpha'_2, & -\alpha'_2 - \alpha'_3, \\ -\alpha'_2 - \alpha'_4, & -\alpha'_1 - \alpha'_2 - \alpha'_3, & -\alpha'_1 - \alpha'_2 - \alpha'_4, \\ -\alpha'_2 - \alpha'_3 - \alpha'_4, & -\alpha'_1 - \alpha'_2 - \alpha'_3 - \alpha'_4, & -\alpha'_1 - 2\alpha'_2 - \alpha'_3 - \alpha'_4. \end{array}$$

Restricting these to T-weights, we find the weights of $\mathfrak{so}_8/\mathfrak{p}'$ are

$$\begin{array}{rcll} -\alpha_2 & = & t_1 - 2t_2 & \text{(once)} \\ -\alpha_1 - \alpha_2 & = & -t_2 & \text{(three times)} \\ -2\alpha_1 - \alpha_2 & = & -t_1 & \text{(three times)} \\ -3\alpha_1 - \alpha_2 & = & -2t_1 + t_2 & \text{(once)} \\ -3\alpha_1 - 2\alpha_2 & = & -t_1 - t_2 & \text{(once),} \end{array}$$

which agree with those of $\mathrm{Hom}(E, \mathrm{End}(E)) \oplus \bigwedge^2 E^*$.

A similar calculation verifies $\mathfrak{g}/\mathfrak{p} \cong \mathrm{Sym}^3 E^* \otimes \bigwedge^2 E \oplus \bigwedge^2 E^*$, and also that the normal space $(\mathfrak{so}_8/\mathfrak{p}')/(\mathfrak{g}/\mathfrak{p})$ has weights

$$\begin{array}{rcll} -\alpha_1 - \alpha_2 & = & -t_2 & \text{(twice)} \\ -2\alpha_1 - \alpha_2 & = & -t_1 & \text{(twice),} \end{array}$$

and hence is isomorphic to $E^* \oplus E^*$ as a $GL(E)$-module. $\square$

APPENDIX C

Graphs and symmetry

Let $\varphi : E \to F$ be a morphism of vector bundles on a variety. There are at least two ways of defining symmetry for such a morphism, with corresponding methods for producing degeneracy loci. The purpose of this appendix is to describe the relationship between these two methods, and to introduce appropriate generalizations. The essential ingredient is a comparison of the exponential map for Lie algebras with the graph map for morphisms.

C.1. The exponential map

In this section, we assume the ground field is $\mathbb{C}$ (but see Remark C.1.3). There is an analytic map $\exp : \mathfrak{gl}_n \to GL_n$ given by

$$\exp(X) = 1 + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \cdots . \tag{C.1.1}$$

The same formula defines an exponential map $\exp : \mathfrak{g} \to G$ for any subgroup $G \subset GL_n$ with Lie algebra $\mathfrak{g}$.

When $N \subset GL_n$ is unipotent, so its Lie algebra $\mathfrak{n}$ is nilpotent, the map $\exp : \mathfrak{n} \to N$ is algebraic. In fact, this is an isomorphism of algebraic varieties, defined over $\mathbb{Z}[\frac{1}{n!}]$.

We shall first consider a special case. Recalling the notation of §A.1, let $P_{\widehat{m}} \subset GL_n$ be a (standard) maximal parabolic, so $GL_n/P_{\widehat{m}} = Gr(m,n)$. Let $N_{\widehat{m}}^-$ be the unipotent radical of the opposite parabolic. Let $\mathfrak{p}_m$ and $\mathfrak{n}_m^-$ be the Lie algebras, so $\mathfrak{gl}_n = \mathfrak{n}_m^- \oplus \mathfrak{p}_m$. Then $X^2 = 0$ for all $X \in \mathfrak{n}_m^-$, so (C.1.1) reduces to $\exp(X) = 1 + X$; in particular, $\exp : \mathfrak{n}_m^- \to N_m^-$ is defined over $\mathbb{Z}$.

Let E be the subspace spanned by the first m standard basis vectors for $V = \mathbb{C}^n$, so it corresponds to the point $[E] = eP_{\widehat{m}}$ in $GL_n/P_{\widehat{m}}$. Let F be the subspace spanned by the last $n-m$ basis vectors. Then $\mathfrak{n}_m^-$ is canonically isomorphic to $\mathrm{Hom}(E,F)$, as can be seen from the block forms of $\mathfrak{p}_m$ and $\mathfrak{n}_m^-$ (using the standard basis to identify $\mathfrak{gl}_n$ with $n \times n$ matrices). Write $\Omega^o = N_{\widehat{m}}^- \cdot P_{\widehat{m}}/P_{\widehat{m}}$ for the open neighborhood of $[E]$ parametrizing subspaces which map surjectively to E under the projection $V = E \oplus F \to E$; as in §A.4, we have $N_{\widehat{m}}^- \cong \Omega^o$ via the map $x \mapsto x \cdot eP_{\widehat{m}}$.

For $\varphi \in \mathrm{Hom}(E,F)$, let $E_\varphi \subset E \oplus F$ be the *graph* of φ, i.e., the subspace $\{(v, \varphi(v)) \,|\, v \in E\}$. Thus there is an injective *graph morphism* $gr : \mathrm{Hom}(E,F) \to Gr(m,n)$ given by $gr(\varphi) = [E_\varphi]$.

Proposition C.1.1. *Under the identifications* $\mathfrak{n}_m^- = \mathrm{Hom}(E,F)$ *and* $N_{\widehat{m}}^- = \Omega^o$, *the map* $\exp : \mathfrak{n}_m^- \to N_{\widehat{m}}^-$ *is identified with the graph morphism* $gr : \mathrm{Hom}(E,F) \to Gr(m,n)$.

Proof. Using the standard basis, the map exp is given in terms of matrices by

$$\left(\begin{array}{c|c} 0_m & 0 \\ \hline * & 0_{n-m} \end{array}\right) \mapsto \left(\begin{array}{c|c} I_m & 0 \\ \hline * & I_{n-m} \end{array}\right),$$

where I_m is the $m \times m$ identity matrix, 0_m is the zero matrix, and $*$ is an $(n-m) \times m$ matrix corresponding to an element $\varphi \in \mathrm{Hom}(E,F)$. The first m columns of the $n \times n$ matrix on the RHS span E_φ. $\square$

Now let $G \subset GL_n$ be a semisimple (or reductive) subgroup such that $P = P_{\widehat{m}} \cap G$ is parabolic in G, so the inclusion induces $i : G/P \hookrightarrow Gr(m,n)$. Let N^- be the unipotent radical of the opposite parabolic of P. Let $\mathfrak{g}, \mathfrak{p}, \mathfrak{n}^-$ be the Lie algebras, considered also as subalgebras of $\mathfrak{gl}_n$. In general, $\mathfrak{n}^-$ is not contained in $\mathfrak{n}_m^-$. (Indeed, this happens exactly when $P_{\widehat{m}}$ is the smallest parabolic subgroup of GL_n containing P.) However, there is a natural inclusion $\mathfrak{g}/\mathfrak{p} \hookrightarrow \mathfrak{gl}_n/\mathfrak{p}_m$, so one obtains an inclusion $\bar{\imath} : \mathfrak{n}^- \hookrightarrow \mathfrak{n}_m^-$ via an isomorphism $\mathfrak{n}^- \cong \mathfrak{g}/\mathfrak{p}$.

Write $\Omega_G^o = N^- \cdot P/P \subset G/P$, and identify this with N^- as before. Thus the exponential map gives an inclusion $\exp : \mathfrak{n}^- \xrightarrow{\sim} N^- = \Omega_G^o \subset \Omega^o \subset Gr(m,n)$.

Proposition C.1.2. *Let* $gr : \mathrm{Hom}(E,F) \to Gr(m,n)$ *be the graph map, and let* $Z \subseteq \mathfrak{n}^-$ *be the subscheme defined by*

$$Z = \{X \in \mathfrak{n}^- \mid X^2 = 0\}, \tag{C.1.2}$$

where "X^2" is defined via the given representation $\mathfrak{n}^- \subset \mathfrak{g} \subset \mathfrak{gl}_n$. *Then* $Z = gr^{-1}(\Omega_G^o) \cap \bar{\imath}(\mathfrak{n}^-)$. *In other words, the diagram*

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, hook] & \mathfrak{n}^- \arrow[d, hook, "gr \circ \bar{\imath}"] \\ \mathfrak{n}^- \arrow[r, hook, "\exp"] & Gr(m,n) \end{tikzcd}$$

is cartesian.

Proof. We want to show $\exp(X) = gr \circ \bar{\imath}(X)$ iff $X \in Z$. Let ν be the smallest integer such that $X^{\nu+1} = 0$ for all $X \in \mathfrak{n}^-$. Then $\exp(X) = 1 + X + \frac{1}{2}X^2 + \cdots + \frac{1}{\nu!}X^\nu$ and $gr \circ \bar{\imath}(X) = 1 + X$, so it is clear that $Z \subseteq gr^{-1}(\Omega_G^o) \cap \bar{\imath}(\mathfrak{n}^-)$.

On the other hand, suppose $\exp(X) = gr \circ \bar{\imath}(X)$, i.e.,

$$\frac{1}{2}X^2 + \cdots + \frac{1}{\nu!}X^\nu = 0. \tag{C.1.3}$$

Since the nonzero powers of a nilpotent element are linearly independent, (C.1.3) implies $X^2 = 0$. $\square$

REMARK C.1.3. The restriction on the characteristic of the ground field is not strictly necessary. The scheme Z of (C.1.2) is defined over $\mathbb{Z}$, as is $\exp|_Z$. The first paragraph of the proof thus shows $Z \subseteq gr^{-1}(\Omega_G^o) \cap \bar{\imath}(\mathfrak{n}^-)$ over any field.

C.2. Two methods

We illustrate two methods for finding formulas for degeneracy loci for a morphism $\varphi : E \to F$, in the case of a symmetric morphism, so $F = E^*$ and $\varphi = \varphi^* : E \to E^*$. Let E have rank m.

C.2.1. Schubert loci in bundles. In the first method, one constructs an auxiliary vector bundle $V = E \oplus E^*$, with a symplectic form ω; specifically, $\omega(x_1 \oplus f_1, x_2 \oplus f_2) = f_1(x_2) - f_2(x_1)$. In other words, V comes with a canonical reduction of structure group from GL_{2m} to Sp_{2m}. The subbundle $E = E \oplus 0 \subset V$ is clearly isotropic, and the morphism φ is symmetric iff its graph E_φ is isotropic in V. Moreover, $\ker(\varphi) = E \cap E_\varphi$, so we have an identification

$$D_r(\varphi) = \Omega_{m-r} := \{x \in X \mid \dim(E(x) \cap E_\varphi(x)) \geq m - r\}.$$

There is an associated Grassmann bundle

$$\mathbf{LG}(V) \xrightarrow{\pi} X,$$

whose fiber over x is the Lagrangian Grassmannian of isotropic m-planes in $V(x)$. Let $S \subset V$ be the tautological rank m subbundle on $\mathbf{LG}(V)$, and let $E \subset V$ also denote the bundles pulled back to $\mathbf{LG}(V)$. In $\mathbf{LG}(V)$, there are *Schubert loci*

$$\boldsymbol{\Omega}_p = \boldsymbol{\Omega}_p(E, S) = \{x \in \mathbf{LG}(V) \,|\, \dim(E(x) \cap S(x)) \geq p\}$$

The bundle $\mathbf{LG}(V) \to X$ has sections s_E and s_φ corresponding to the isotropic subbundles E and E_φ, and one sees that $D_r(\varphi) = s_\varphi^{-1}(\boldsymbol{\Omega}_{m-r})$:

$$\begin{tikzcd} \boldsymbol{\Omega}_{m-r} \arrow[r, hook] & \mathbf{LG}(V) \\ D_r(\varphi) \arrow[u] \arrow[r, hook] & X. \arrow[u, "s_\varphi"'] \end{tikzcd}$$

Thus the problem is reduced to computing formulas for Schubert loci in $H^*\mathbf{LG}(V)$. This can be done by first finding a formula for the most degenerate locus, $\boldsymbol{\Omega}_m = \{x \,|\, E(x) = S(x)\}$, and then applying *divided difference operators* to deduce formulas for the others.

This is essentially the method used by Fulton in [**Fu2**] and [**Fu3**] (see also [**Fu-Pr**]). An advantage of this approach is that it works for all flag bundles, and the machinery of divided difference operators reduces the problem (in principle) to that of computing a single class for each case.

C.2.2. Orbits. An alternative approach is to work with the space of symmetric morphisms more directly. Continuing with the example of a symmetric morphism $\varphi : E \to E^*$, one proceeds as follows. Note that φ corresponds to a section s_φ of the vector bundle $\mathrm{Sym}^2 E^* \to X$. Inside this vector bundle there is a locus $\mathbf{D}_r$ of morphisms with rank at most r; we are looking for the locus where s_φ meets $\mathbf{D}_r$:

$$\begin{tikzcd} \mathbf{D}_r \arrow[r, hook] & \mathrm{Sym}^2 E^* \\ D_r(\varphi) \arrow[u] \arrow[r, hook] & X. \arrow[u, "s_\varphi"'] \end{tikzcd}$$

It suffices to solve this on the classifying space for E, so take $X = BGL_m$. Then $\mathrm{Sym}^2 E^*$ is the vector bundle corresponding to the representation $\mathrm{Sym}^2(\mathbb{C}^m)^*$ of GL_m. This representation has finitely many GL_m-orbits O_r, for $0 \leq r \leq m$, corresponding to the symmetric matrices of rank r. Let $D_r = \overline{O}_r$ be the locus of matrices of rank at most r, so we have

$$\mathbf{D}_r = EGL_m \times^{GL_m} D_r \subseteq EGL_m \times^{GL_m} \mathrm{Sym}^2(\mathbb{C}^m)^* = \mathrm{Sym}^2 E^*$$

over BGL_m. (Here $EGL_m \to BGL_m$ is the universal principal GL_m-bundle.)

Computing the class $[D_r(\varphi)] \in H^*X$ thus reduces to computing the *equivariant* class of an orbit closure in a vector space: we want a formula for $[D_r]^G \in H^*_G(\mathrm{Sym}^2(\mathbb{C}^m)^*) = H^*_G(pt)$. This is essentially the method used by Harris and Tu [**Ha-Tu**] (without the language of equivariant cohomology), and developed further by Fehér, Némethi, and Rimányi [**Fe-Né-Ri**]; see also [**Fe-Ri2**].

This method has the advantage of producing manifestly *unique* formulas: the equivariant cohomology ring $H^*_G(pt)$ is frequently a polynomial ring. (In fact, it will be a polynomial ring for the cases we study.) Also, it is often an easy matter to determine the most degenerate class $[D_0]^G$. Using an embedding $H^*_G(pt) \subset H^*_T(pt)$, for T a maximal torus, the orbit $D_0 = \{0\}$ has class equal to the product of the T-weights of the representation. On the other hand, it is less clear what the action of divided difference operators means in this context; they may not produce formulas for orbit closures.

C.3. General setup

Here we introduce two general notions of symmetry for morphisms corresponding to the two methods, called "G/P symmetry" and "$\mathfrak{g}/\mathfrak{p}$ symmetry," respectively. The example discussed above will correspond to the case $G = Sp_{2m}$, with maximal parabolic $P = P_{\widehat{m}}$, so $G/P = LG(2m)$; the two notions coincide in this case.

Given a map of vector bundles $\varphi : E_m \to F_{n-m}$ on X, let $V = E \oplus F$, so V has rank n. From this data, we obtain a (not commutative!) diagram:

$$\begin{tikzcd} Hom(E,F) \arrow[dr, hook, "gr"] & \\ X \arrow[u, "s_\varphi"] \arrow[r, "s_E"'] & \mathbf{Gr}(m,V). \end{tikzcd}$$

Here s_φ and s_E are the sections determined by $\varphi : E \to F$ and $E \subset V$, respectively, and gr is the **graph morphism**, defined locally as in §C.1.[1] Injectivity is easily checked on fibers: for a point $x \in X$, $gr(x)$ gives an isomophism from $\mathrm{Hom}(E(x), F(x))$ onto the open Schubert cell in $Gr(m, V(x))$ consisting of subspaces projecting isomorphically to $E(x)$. Note that $gr \circ s_\varphi = s_{E_\varphi}$ is the section corresponding to the graph of φ.

Since $Hom(S,Q)$ is the relative tangent bundle to $\mathbf{Gr}(m,V) \to X$, we see that $Hom(E,F) = s_E^* Hom(S,Q)$ is the normal bundle to $s_E(X) \subset \mathbf{Gr}(m,V)$, and the map gr gives a tubular neighborhood of $s_E(X)$ in $\mathbf{Gr}(m,V)$.

The subbundle $E \subset V$ corresponds to a reduction of structure group from GL_n to a maximal parabolic subgroup $P_{\widehat{m}}$, and the splitting $V = E \oplus F$ corresponds to a further reduction to a Levi subgroup $L_m \cong GL_m \times GL_{n-m} \subset P_{\widehat{m}}$. (Here we have implicitly chosen a maximal torus and assumed $P_{\widehat{m}}$ is standard.) Write $\mathbf{L}_m \subset \mathbf{P}_{\widehat{m}} \subset \mathbf{GL}_n$ for the corresponding principal bundles on X.

Suppose V also admits a reduction of structure group to a reductive subgroup $G \subset GL_n$, such that the stabilizer of E (in a fiber) is a maximal parabolic subgroup $P \subset G$. Fix such a G and P, so $P = G \cap P_{\widehat{m}}$, and $L = L_m \cap G$ is a Levi subgroup of P. We have corresponding principal bundles[2]

$$\begin{tikzcd} \mathbf{L}_m \arrow[r, hook] & \mathbf{P}_{\widehat{m}} \arrow[r, hook] & \mathbf{GL}_n \\ \mathbf{L} \arrow[u, hook] \arrow[r, hook] & \mathbf{P} \arrow[u, hook] \arrow[r, hook] & \mathbf{G}. \arrow[u, hook] \end{tikzcd}$$

Taking the quotient of $\mathbf{G} \to X$ by the right action of P, we obtain a G/P-bundle $\pi : \mathbf{G}/\mathbf{P} \to X$, with a closed inclusion $\mathbf{G}/\mathbf{P} \hookrightarrow \mathbf{Gr}(m,V)$ over X. By construction, the subbundle $E \subset V$ defines a section of π, which we shall also denote by s_E.

The relative tangent bundle to π is

$$T_\pi = \underline{\mathfrak{g}/\mathfrak{p}} := \mathbf{G} \times^P \mathfrak{g}/\mathfrak{p}.$$

[1] Equivalently, one can define gr as follows. Consider the tautological map $\Phi : E \to F$ on $Hom(E,F)$. Then gr is given by the graph subbundle $E_\Phi \subset V$ (and the universal property of $\mathbf{Gr}(m,V)$).

[2] These principal bundles may not be locally trivial in the Zariski topology, but only in the étale topology. When G is *special* (in the semisimple case, a product of factors of type SL_n or Sp_{2n}), all principal bundles are Zariski-locally trivial.

Thus $\underline{\mathfrak{g}/\mathfrak{p}} \subset Hom(S,Q)$ as bundles on $\mathbf{G}/\mathbf{P}$, and $s_E^*\underline{\mathfrak{g}/\mathfrak{p}} \subset Hom(E,F)$ as bundles on X.

Definition C.3.1. A morphism $\varphi : E \to F$ is G/P**-symmetric** if its graph defines a section of $\mathbf{G}/\mathbf{P} \subseteq \mathbf{Gr}(m,V)$. A morphism φ is $\mathfrak{g}/\mathfrak{p}$**-symmetric** if s_φ is a section of the subbundle $s_E^*\underline{\mathfrak{g}/\mathfrak{p}} \subseteq Hom(E,F)$.

Equivalently, φ is G/P-symmetric if s_φ lies in $gr^{-1}\mathbf{G}/\mathbf{P} \subseteq \mathrm{Hom}(E,F)$, and $\mathfrak{g}/\mathfrak{p}$-symmetric if s_φ lies in $\Gamma(X, s_E^*\underline{\mathfrak{g}/\mathfrak{p}}) \subseteq \mathrm{Hom}(E,F)$. The latter is a linear subspace of $\mathrm{Hom}(E,F)$, while the former is not linear in general.

Example C.3.2. A symmetric morphism $E \to E^*$ is $Sp_{2m}/P_{\widehat{m}}$-symmetric, as well as $\mathfrak{sp}_{2m}/\mathfrak{p}_m$-symmetric. Any morphism $\varphi : E_m \to F_{n-m}$ is $GL_n/P_{\widehat{m}}$-symmetric and $\mathfrak{gl}_n/\mathfrak{p}_m$-symmetric.

Example C.3.3. A triality-symmetric morphism (§1.1.7) is $\mathfrak{g}_2/\mathfrak{p}$-symmetric, for $\mathfrak{p}$ corresponding to the long root. Triality-symmetric morphisms are not in general G_2/P-symmetric.

The relation between these two notions is as follows. On $\mathbf{G}/\mathbf{P}$, there is an exact sequence of vector bundles

$$0 \to \underline{\mathfrak{p}} \to \underline{\mathfrak{g}} \to \underline{\mathfrak{g}/\mathfrak{p}} \to 0,$$

and this splits on X when pulled back via s_E. Indeed, $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{n}^-$ as L-modules, so we have

$$s_E^*\underline{\mathfrak{g}/\mathfrak{p}} \cong \mathbf{L} \times^L \mathfrak{g}/\mathfrak{p} \cong \mathbf{L} \times^L \mathfrak{n}^- \hookrightarrow s_E^*\underline{\mathfrak{g}}.$$

Composing this with the inclusion $s_E^*\underline{\mathfrak{g}} \hookrightarrow s_E^*\underline{\mathfrak{gl}}_n = End(V)$, we have an inclusion

$$s_E^*\underline{\mathfrak{g}/\mathfrak{p}} \hookrightarrow End(V).$$

(In general this does not factor through $Hom(E,F)$!) We use this inclusion to define the **square** of a section s_φ of $s_E^*\underline{\mathfrak{g}/\mathfrak{p}}$.

Proposition C.3.4. *As a subscheme of $s_E^*\underline{\mathfrak{g}/\mathfrak{p}} \subseteq Hom(E,F)$, the intersection $\mathbf{Z} = s_E^*\underline{\mathfrak{g}/\mathfrak{p}} \cap gr^{-1}\mathbf{G}/\mathbf{P}$ is defined by the vanishing of the square of a generic section. That is, locally over X, the equations are*

$$\mathbf{Z} = \{(x,\varphi) \,|\, \varphi(x)^2 \equiv 0\}.$$

Proof. The statement is local on X, and Proposition C.1.2 says that $\varphi(x)$ is G/P-symmetric iff its square (with respect to the representation $\mathfrak{n}^- \hookrightarrow \mathfrak{gl}_n = \mathrm{End}(V(x))$) is zero. □

Corollary C.3.5. *Let $\varphi : E \to F$ be a $\mathfrak{g}/\mathfrak{p}$-symmetric morphism, corresponding to a section s_φ of $s_E^*\underline{\mathfrak{g}/\mathfrak{p}}$. Let $Z_\varphi \subseteq X$ be the subscheme defined by*

$$Z_\varphi = s_\varphi^{-1}\mathbf{Z}. \tag{C.3.1}$$

Then Z_φ is the zero locus of the section $(s_\varphi)^2$ of $End(V)$. Write $\varphi|_Z : E|_Z \to F|_Z$ for the morphism of vector bundles restricted to Z_φ. Then $\varphi|_Z$ is G/P-symmetric, and Z_φ is the largest subscheme of X with this property.

APPENDIX D

Tables

D.1. Parametrizations of Schubert cells

These parametrizations are given in terms of the standard γ-isotropic basis for V; see (2.2.6). Lowercase variables are free; capital variables are determined by the equations in the right-hand column.

$$\Omega^o_{id} = X^o_{7\,6} = \begin{pmatrix} X & a & b & c & d & e & 1 \\ Y & Z & S & T & f & 1 & 0 \end{pmatrix} \qquad \begin{aligned} X &= -ae - bd - c^2 \\ Y &= -a - bf + cd - cef \\ Z &= -cf - d^2 + def \\ S &= c + de - e^2 f \\ T &= -d + ef \end{aligned}$$

$$\Omega^o_{t} = X^o_{7\,5} = \begin{pmatrix} X & a & b & c & d & e & 1 \\ Y & S & Z & T & 1 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} X &= -ae - bd - c^2 \\ Y &= -b - ce \\ Z &= -e^2 \\ S &= -c + de \\ T &= e. \end{aligned}$$

$$\Omega^o_{s} = X^o_{6\,7} = \begin{pmatrix} a & X & b & c & d & 1 & 0 \\ Y & Z & e & S & T & 0 & 1 \end{pmatrix} \qquad \begin{aligned} X &= -bd - c^2 \\ Y &= -b^2 + ce \\ Z &= -a - bc - de \\ S &= b \\ T &= -c \end{aligned}$$

$$\Omega^o_{st} = X^o_{6\,3} = \begin{pmatrix} a & X & b & c & d & 1 & 0 \\ S & Y & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} X &= -bd - c^2 \\ Y &= -d \\ S &= c \end{aligned}$$

$$\Omega^o_{ts} = X^o_{5\,7} = \begin{pmatrix} a & b & X & c & 1 & 0 & 0 \\ Y & d & Z & S & 0 & T & 1 \end{pmatrix} \qquad \begin{aligned} X &= -c^2 \\ Y &= -b^2 - cd \\ Z &= -a + bc \\ S &= -b \\ T &= c \end{aligned}$$

$$\Omega^o_{tst} = X^o_{5\,2} = \begin{pmatrix} a & b & X & c & 1 & 0 & 0 \\ S & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} X &= -c^2 \\ S &= -c \end{aligned}$$

$$\Omega^o_{sts} = X^o_{3\,6} = \begin{pmatrix} a & b & 1 & 0 & 0 & 0 & 0 \\ c & X & 0 & S & Y & 1 & 0 \end{pmatrix} \qquad \begin{aligned} X &= -a^2 \\ Y &= -b \\ S &= a \end{aligned}$$

$$\Omega^o_{stst} = X^o_{3\,1} = \begin{pmatrix} a & b & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Omega^o_{tsts} = X^o_{2\,5} = \begin{pmatrix} a & 1 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & X & S & 1 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} X &= -a^2 \\ S &= -a \end{aligned}$$

$$\Omega^o_{tstst} = X^o_{2\,1} = \begin{pmatrix} a & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Omega^o_{ststs} = X^o_{1\,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Omega^o_{w_0} = X^o_{1\,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

D.2. Formulas

Here $P_w = P_w(x;y;v) = [\mathbf{\Omega}_w]$ in $H^*\mathbf{Fl}_\gamma(V)$.

$$\begin{aligned}P_{w_0} = \frac{1}{2}(&x_1^3 - 2\,x_1^2\,y_1 + x_1\,y_1^2 - x_1\,y_2^2 + x_1\,y_1\,y_2 - y_1^2\,y_2 + y_1\,y_2^2\\ &+ 5\,x_1^2\,v - 7\,x_1\,y_1\,v + x_1\,y_2\,v + 2\,y_1^2\,v + y_1\,y_2\,v - 2\,y_2^2\,v\\ &+ 8\,x_1\,v^2 - 6\,y_1\,v^2 + 2\,y_2\,v^2 + 4\,v^3)\\ \times\,&(x_1^2 + x_1\,y_1 + y_1\,y_2 - y_2^2 + x_1\,v + y_2\,v)(x_2 - x_1 - y_2 + v)\end{aligned}$$

$$\begin{aligned}P_{tstst} = -\frac{1}{2}(&14\,y_1y_2^2v^2 + 6\,x_1x_2^2v^2 - x_1x_2^3y_2 + x_1y_1^2y_2^2 + x_1^2y_1v^2 + 13\,x_1x_2v^3\\ &- 6\,y_1^3y_2v - x_1^3y_1y_2 + 10\,y_1y_2v^3 - 4\,x_1^2y_1y_2^2 - x_1y_1^3y_2 - x_1y_1^3v\\ &+ 2\,x_1x_2y_2^3 + x_1^2x_2^2v - 14\,x_1y_2v^3 + 3\,x_1^3y_1v + 11\,y_1^2y_2^2v - 6\,y_1y_2^3v\\ &- 7\,x_1y_2^2v^2 + 6\,y_1^2y_2v^2 - 3\,x_1y_1v^3 + 10\,x_1y_2^3v + 3\,x_1^2y_1^2v + 6\,x_1^2x_2v^2\\ &- 8\,x_1^3y_2v - 19\,x_1^2y_2v^2 + 2\,x_2^3y_2^2 + 3\,x_1^2y_1^2y_2 + x_1x_2^3v + 4\,x_1^2y_2^2v\\ &- x_2y_1^3v + 6\,x_1x_2y_1y_2v + x_2^4y_1 - 5\,x_1^4v - 7\,x_1^3v^2 + x_1^2v^3 + 8\,x_1v^4\\ &- x_1^5 - x_2^5 - 5\,x_2^4v - 7\,x_2^3v^2 + x_2^2v^3 + 8\,x_2v^4 + 4\,v^5 + x_1^3x_2v\\ &+ 8\,x_1y_1^2y_2v - 12\,x_1y_1y_2^2v + 7\,x_1y_1y_2v^2 + x_1^4y_1 + 2\,x_1^3y_2^2 + x_1^3y_1^2\\ &- x_1^2y_1^3 - x_1y_2^4 + 2\,y_1^3y_2^2 - 5\,y_1^2y_2^3 + 4\,y_1y_2^4 - y_2^4v + 15\,y_2^3v^2 - 15\,y_2^2v^3\\ &- 2\,y_2v^4 - 3\,x_2y_1v^3 - x_2^3y_1y_2 - 8\,x_2^3y_2v + 3\,x_2^3y_1v + 3\,x_2^2y_1^2y_2\\ &- 4\,x_2^2y_1y_2^2 + 4\,x_2^2y_2^2v - 19\,x_2^2y_2v^2 + 3\,x_2^2y_1^2v - x_2y_1^3y_2 + x_2y_1^2y_2^2\\ &+ x_2^2y_1v^2 + 10\,x_2y_2^3v - 7\,x_2y_2^2v^2 - 14\,x_2y_2v^3 + x_2^3y_1^2 - x_2^2y_1^3 - x_2y_2^4\\ &+ 8\,x_2y_1^2y_2v - 12\,x_2y_1y_2^2v + 7\,x_2y_1y_2v^2 - 4\,y_1^2v^3 + 2\,y_1^3v^2 - 2\,y_1v^4\\ &+ x_1^2x_2y_1y_2 - 6\,x_1^2x_2y_2v - x_1^2x_2y_1v + x_1x_2^2y_1y_2 - 6\,x_1x_2^2y_2v\\ &- 4\,x_1x_2y_1v^2 - 2\,x_1x_2y_1y_2^2 - x_1x_2y_1^2v - 11\,x_1x_2y_2v^2 + x_1x_2y_1^2y_2\\ &- 4\,x_1x_2y_2^2v - x_1x_2^2y_1v - y_2^5 - x_2^4y_2 + 2\,x_2^2y_2^3 - x_1^4y_2 + 2\,x_1^2y_2^3\\ &- x_1^2x_2^2y_2 - x_1^3x_2y_2)\end{aligned}$$

$$\begin{aligned}P_{ststs} = \frac{1}{2}&(x_1 - y_1 + 2\,v)(x_1 - y_2 + 2\,v)(x_1 - y_1 + y_2 + v)\\ &\times(x_1 + y_1 - y_2 + v)(x_1 + y_2)\end{aligned}$$

$$
\begin{aligned}
P_{tsts} = -\frac{1}{2}(&14\,x_1^2y_2v + 4\,x_1y_2^3 - 4\,x_1^3y_2 + 12\,x_1v^3 - 2\,y_2v^3 - 9\,y_2^2v^2 \\
&+ 11\,x_1^2v^2 + 8\,y_2^3v - 2\,y_1v^3 - x_1y_1^3 + x_1^3y_1 + y_1^2y_2^2 + 7\,x_1y_1y_2v \\
&- y_1^3y_2 + 4\,x_1y_1^2y_2 + 2\,x_1x_2^2y_1 - 2\,x_1x_2y_2^2 - x_1x_2y_1^2 - x_1x_2^2v \\
&- 3\,x_1x_2^2y_2 - 2\,x_1^2x_2y_1 + 3\,x_1^2x_2y_2 - x_2^2y_1y_2 + x_2y_1v^2 \\
&- 6\,x_2y_2v^2 + 2\,x_2y_2^2v + x_2^2y_1v - 6\,x_2^2y_2v + x_2y_1^2v - x_2y_1y_2v \\
&- 6\,x_1y_1y_2^2 - 5\,x_1y_1v^2 - 4\,x_1y_2^2v + 2\,x_1^3v - 12\,x_1y_2v^2 \\
&+ x_1x_2y_1v - y_2^4 + 5\,y_1^2y_2v - 3\,x_1x_2v^2 + 2\,x_1^2y_2^2 + 2\,x_2^2y_2^2 \\
&+ x_1x_2y_1y_2 - 8\,y_1y_2^2v + x_1^2y_1^2 - 2\,x_1^2y_1v + x_1^2y_1y_2 + 4\,v^4 \\
&- x_2^2v^2 - 2\,x_2^3v - x_1^4 + 3\,x_1x_2y_2v + x_2^2y_1^2 - 2\,y_1^2v^2 - x_2^4 \\
&+ 3\,x_1^3x_2 - 4\,x_1^2x_2^2 + 2\,x_1x_2^3 + 6\,y_1y_2v^2)
\end{aligned}
$$

$$
\begin{aligned}
P_{stst} = \frac{1}{2}(&2\,x_1^2y_2v + 12\,x_1v^3 + 10\,y_2v^3 - 9\,y_2^2v^2 + 13\,x_1^2v^2 - 2\,y_2^3v \\
&- 2\,y_1v^3 + x_1y_1^3 - x_1^3y_1 + y_1^2y_2^2 + 2\,y_1^3v + 6\,x_1y_1y_2v \\
&- 2\,x_1y_1^2y_2 + 6\,x_1^2x_2v - x_1x_2^2y_1 - 2\,x_1x_2y_2^2 - x_1x_2y_1^2 \\
&+ 6\,x_1x_2^2v - x_1^2x_2y_1 + 2\,x_2y_1y_2^2 - 2\,x_2y_1^2y_2 + 2\,x_2^2y_1y_2 \\
&- 5\,x_2y_1v^2 + 8\,x_2y_2v^2 - 8\,x_2y_2^2v - 4\,x_2^2y_1v + 2\,x_2^2y_2v \\
&- 4\,x_2y_1^2v + 6\,x_2y_1y_2v - 4\,x_1y_1^2v + 2\,x_1y_1y_2^2 - 5\,x_1y_1v^2 \\
&- 8\,x_1y_2^2v + 6\,x_1^3v + 8\,x_1y_2v^2 - 4\,x_1x_2y_1v + y_2^4 - 6\,y_1^2y_2v \\
&+ 13\,x_1x_2v^2 - 2\,x_1^2y_2^2 - x_2^3y_1 - 2\,x_2^2y_2^2 + 2\,x_1x_2y_1y_2 + 8\,y_1y_2^2v \\
&- x_1^2y_1^2 + x_2y_1^3 - 2\,y_1y_2^3 - 4\,x_1^2y_1v + 2\,x_1^2y_1y_2 + 12\,x_2v^3 + 4\,v^4 \\
&+ 13\,x_2^2v^2 + 6\,x_2^3v + x_1^4 + 2\,x_1x_2y_2v - x_2^2y_1^2 - 4\,y_1^2v^2 + x_2^4 \\
&+ x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + 4\,y_1y_2v^2)
\end{aligned}
$$

$$
\begin{aligned}
P_{tst} = -\frac{1}{2}(&4\,x_1^2y_2 + 4\,x_1^2v + 12\,x_1v^2 + 8\,y_1y_2v + 12\,v^3 + 2\,x_1x_2y_2 \\
&+ 5\,x_1x_2v - 3\,x_1x_2y_1 + 2\,x_2y_1y_2 + 2\,x_1y_1y_2 - 8\,x_2y_2v \\
&- 3\,x_2y_1v - 8\,x_1y_2v - 3\,x_1y_1v - y_1^3 + 4\,x_2^2v + 12\,x_2v^2 \\
&- 4\,x_2^2y_2 + x_2^2y_1 + x_1x_2^2 + x_1^2x_2 + x_1^2y_1 + 4\,y_1^2y_2 \\
&- y_1^2v - 6\,y_1y_2^2 - 6\,y_1v^2 - 6\,y_2^2v - 6\,y_2v^2 + 4\,y_2^3)
\end{aligned}
$$

$$
\begin{aligned}
P_{sts} = \frac{1}{2}(&4\,x_1^3 + 12\,x_1^2 v + 12\,x_1 v^2 + 4\,y_1 y_2 v + 4\,v^3 - x_1 x_2 v \\
&+ x_1 x_2 y_1 + 4\,x_1 y_1 y_2 - x_2 y_1 v + 4\,x_1 y_2 v - 5\,x_1 y_1 v \\
&+ y_1^3 + 4\,x_2^2 v + 4\,x_2 v^2 - x_2^2 y_1 + 3\,x_1 x_2^2 - 3\,x_1^2 x_2 \\
&- 3\,x_1^2 y_1 - 2\,x_1 y_1^2 - 4\,x_1 y_2^2 - 2\,y_1^2 y_2 - 3\,y_1^2 v \\
&+ 2\,y_1 y_2^2 - 2\,y_1 v^2 - 6\,y_2^2 v + 6\,y_2 v^2)
\end{aligned}
$$

$$
P_{ts} = (x_1 - y_2 + 2\,v)(x_1 - y_1 + 2\,v)
$$

$$
\begin{aligned}
P_{st} = &2\,x_1^2 - x_1 x_2 - x_1 y_1 + 4\,x_1 v + 2\,x_2^2 - x_2 y_1 + 4\,x_2 v \\
&- y_1^2 + 2\,y_1 y_2 - 2\,y_1 v - 2\,y_2^2 + 2\,y_2 v + 4\,v^2
\end{aligned}
$$

$$
P_t = x_1 + x_2 - y_1 - y_2 + 4\,v
$$

$$
P_s = x_1 - y_1 + 2\,v
$$

$$
P_{id} = 1
$$

D.3. Localizations of equivariant Schubert classes

This table gives the restrictions of equivariant Schubert classes to fixed points, computed as $\sigma_w|_v = P_w(v(t);t)$. (Here $v \in W$ acts on the torus weights via the embedding $W \hookrightarrow S_7$ and the identification $(t_1, t_2, \ldots, t_7) = (t_1, t_2, t_1 - t_2, 1, t_2 - t_1, -t_2, -t_1)$.) We only list the nonzero restrictions, i.e., $\sigma_w|_v$ for $v \geq w$.

It is easy to read the singular loci of Schubert varieties from these formulas. By a theorem of Kumar (cf. [**Bi-La**, §7.2]), vB is a nonsingular point of Ω_w iff $\sigma_w|_v$ is a product of roots, and it is a rationally smooth point iff $\sigma_w|_v$ is an integral multiple of a product of roots. We see that all G_2 Schubert varieties are rationally smooth, and

Ω_{tst} is singular along Ω_{tstst};
Ω_{ts} is singular along Ω_{tsts};
Ω_{st} is singular along Ω_{stst};
Ω_{s} is singular along Ω_{ststs};
Ω_{t} is singular along Ω_{tst}.

$$\sigma_{w_0}|_{w_0} = t_1 t_2 (t_1 - t_2)(t_1 + t_2)(2t_1 - t_2)(-t_1 + 2t_2)$$

$$\begin{aligned}
\sigma_{ststs}|_{w_0} &= t_1 t_2 (t_1 - t_2)(t_1 + t_2)(-2t_1 + t_2) \\
\sigma_{ststs}|_{ststs} &= t_1 t_2 (t_1 - t_2)(t_1 + t_2)(-2t_1 + t_2)
\end{aligned}$$

$$\begin{aligned}
\sigma_{tstst}|_{w_0} &= t_1 t_2 (t_1 + t_2)(2t_1 - t_2)(t_1 - 2t_2) \\
\sigma_{tstst}|_{tstst} &= t_1 t_2 (t_1 + t_2)(2t_1 - t_2)(t_1 - 2t_2)
\end{aligned}$$

$$\begin{aligned}
\sigma_{tsts}|_{w_0} &= t_1 t_2 (t_1 + t_2)(2t_1 - t_2) \\
\sigma_{tsts}|_{ststs} &= t_1 t_2 (t_1 + t_2)(2t_1 - t_2) \\
\sigma_{tsts}|_{tstst} &= t_1 t_2 (t_1 + t_2)(-t_1 + 2t_2) \\
\sigma_{tsts}|_{tsts} &= t_1 t_2 (t_1 + t_2)(-t_1 + 2t_2)
\end{aligned}$$

$$\begin{aligned}
\sigma_{stst}|_{w_0} &= t_1 t_2 (t_1 + t_2)(2t_1 - t_2) \\
\sigma_{stst}|_{ststs} &= t_1 (t_1 - t_2)(t_1 + t_2)(2t_1 - t_2) \\
\sigma_{stst}|_{tstst} &= t_1 t_2 (t_1 + t_2)(2t_1 - t_2) \\
\sigma_{stst}|_{stst} &= t_1 (t_1 - t_2)(t_1 + t_2)(2t_1 - t_2)
\end{aligned}$$

$$\begin{aligned}
\sigma_{sts}|_{w_0} &= t_1(t_1+t_2)(-2t_1+t_2)\\
\sigma_{sts}|_{ststs} &= t_1(t_1+t_2)(-2t_1+t_2)\\
\sigma_{sts}|_{tstst} &= -t_1t_2(t_1+t_2)\\
\sigma_{sts}|_{tsts} &= -t_1t_2(t_1+t_2)\\
\sigma_{sts}|_{stst} &= -t_1(t_1-t_2)(2t_1-t_2)\\
\sigma_{sts}|_{sts} &= -t_1(t_1-t_2)(2t_1-t_2)
\end{aligned}$$

$$\begin{aligned}
\sigma_{tst}|_{w_0} &= -3t_1t_2(t_1+t_2)\\
\sigma_{tst}|_{ststs} &= -t_1(t_1+t_2)(2t_1-t_2)\\
\sigma_{tst}|_{tstst} &= -3t_1t_2(t_1+t_2)\\
\sigma_{tst}|_{tsts} &= -t_2(t_1+t_2)(-t_1+2t_2)\\
\sigma_{tst}|_{stst} &= -t_1(t_1+t_2)(2t_1-t_2)\\
\sigma_{tst}|_{tst} &= -t_2(t_1+t_2)(-t_1+2t_2)
\end{aligned}$$

$$\begin{aligned}
\sigma_{ts}|_{w_0} &= 2t_1(t_1+t_2)\\
\sigma_{ts}|_{ststs} &= 2t_1(t_1+t_2)\\
\sigma_{ts}|_{tstst} &= 2t_2(t_1+t_2)\\
\sigma_{ts}|_{tsts} &= 2t_2(t_1+t_2)\\
\sigma_{ts}|_{stst} &= t_1(2t_1-t_2)\\
\sigma_{ts}|_{sts} &= t_1(2t_1-t_2)\\
\sigma_{ts}|_{tst} &= t_2(-t_1+2t_2)\\
\sigma_{ts}|_{ts} &= t_2(-t_1+2t_2)
\end{aligned}$$

$$\begin{aligned}
\sigma_{st}|_{w_0} &= 2t_1(t_1+t_2)\\
\sigma_{st}|_{ststs} &= 2t_1(2t_1-t_2)\\
\sigma_{st}|_{tstst} &= 2t_1(t_1+t_2)\\
\sigma_{st}|_{tsts} &= t_2(t_1+t_2)\\
\sigma_{st}|_{stst} &= 2t_1(2t_1-t_2)\\
\sigma_{st}|_{sts} &= (t_1-t_2)(2t_1-t_2)\\
\sigma_{st}|_{tst} &= t_2(t_1+t_2)
\end{aligned}$$

$$\sigma_{st}|_{st} = (t_1 - t_2)(2t_1 - t_2)$$

$$\sigma_s|_{w_0} = -2t_1$$
$$\sigma_s|_{ststs} = -2t_1$$
$$\sigma_s|_{tstst} = -(t_1 + t_2)$$
$$\sigma_s|_{tsts} = -(t_1 + t_2)$$
$$\sigma_s|_{stst} = -(2t_1 - t_2)$$
$$\sigma_s|_{sts} = -(2t_1 - t_2)$$
$$\sigma_s|_{tst} = -t_2$$
$$\sigma_s|_{ts} = -t_2$$
$$\sigma_s|_{st} = -(t_1 - t_2)$$
$$\sigma_s|_{s} = -(t_1 - t_2)$$

$$\sigma_t|_{w_0} = -2(t_1 + t_2)$$
$$\sigma_t|_{ststs} = -3t_1$$
$$\sigma_t|_{tstst} = -2(t_1 + t_2)$$
$$\sigma_t|_{tsts} = -3t_2$$
$$\sigma_t|_{stst} = -3t_1$$
$$\sigma_t|_{sts} = -(2t_1 - t_2)$$
$$\sigma_t|_{tst} = -3t_2$$
$$\sigma_t|_{ts} = t_1 - 2t_2$$
$$\sigma_t|_{st} = -(2t_1 - t_2)$$
$$\sigma_t|_{t} = t_1 - 2t_2$$

D.4. Multiplication table of $H_T^*(Fl_\gamma, \mathbb{Z})$

Let $\sigma_w = [\Omega_w]^T$, and use generators $s = s_{\alpha_1}$ and $t = s_{\alpha_2}$. The coefficients can be calculated by using the localization formulas for $\sigma_w|_v$ given in §D.3. Computations of equivariant products are also given in [**Gri-Ram**, §5]. Some translation of notation is required: to obtain theirs from ours, set $\alpha_{pq} = p(t_2 - t_1) + q(t_1 - 2t_2)$, $s_1 = s$, $s_2 = t$, and $[w] = [X_w] = \sigma_{w\,w_0}$.

$$\begin{aligned}
\sigma_s \cdot \sigma_s &= \sigma_{ts} + (t_2 - t_1)\,\sigma_s \\
\sigma_s \cdot \sigma_t &= \sigma_{ts} + \sigma_{st} \\
\sigma_s \cdot \sigma_{ts} &= 2\,\sigma_{sts} + (-t_2)\,\sigma_{ts} \\
\sigma_s \cdot \sigma_{st} &= \sigma_{sts} + \sigma_{tst} + (t_2 - t_1)\,\sigma_{st} \\
\sigma_s \cdot \sigma_{sts} &= \sigma_{tsts} + (t_2 - 2t_1)\,\sigma_{sts} \\
\sigma_s \cdot \sigma_{tst} &= \sigma_{tsts} + 2\,\sigma_{stst} + (-t_2)\,\sigma_{tst} \\
\sigma_s \cdot \sigma_{tsts} &= \sigma_{ststs} + (-t_1 - t_2)\,\sigma_{tsts} \\
\sigma_s \cdot \sigma_{stst} &= \sigma_{ststs} + \sigma_{tstst} + (t_2 - 2t_1)\,\sigma_{stst} \\
\sigma_s \cdot \sigma_{ststs} &= 2(-t_1)\,\sigma_{ststs} \\
\sigma_s \cdot \sigma_{tstst} &= \sigma_{w_0} + (-t_1 - t_2)\,\sigma_{tstst} \\
\sigma_s \cdot \sigma_{w_0} &= 2(-t_1)\,\sigma_{w_0}
\end{aligned}$$

$$\begin{aligned}
\sigma_t \cdot \sigma_t &= 3\,\sigma_{st} + (t_1 - 2t_2)\,\sigma_t \\
\sigma_t \cdot \sigma_{ts} &= 3\,\sigma_{sts} + \sigma_{tst} + (t_1 - 2t_2)\,\sigma_{ts} \\
\sigma_t \cdot \sigma_{st} &= 2\,\sigma_{tst} + (t_2 - 2t_1)\,\sigma_{st} \\
\sigma_t \cdot \sigma_{sts} &= 2\,\sigma_{tsts} + \sigma_{stst} + (t_2 - 2t_1)\,\sigma_{sts} \\
\sigma_t \cdot \sigma_{tst} &= 3\,\sigma_{stst} + 3(-t_2)\,\sigma_{tst} \\
\sigma_t \cdot \sigma_{tsts} &= 3\,\sigma_{ststs} + \sigma_{tstst} + 3(-t_2)\,\sigma_{tsts} \\
\sigma_t \cdot \sigma_{stst} &= \sigma_{tstst} + 3(-t_1)\,\sigma_{stst} \\
\sigma_t \cdot \sigma_{ststs} &= \sigma_{w_0} + 3(-t_1)\,\sigma_{ststs} \\
\sigma_t \cdot \sigma_{tstst} &= 2(-t_1 - t_2)\,\sigma_{tstst} \\
\sigma_t \cdot \sigma_{w_0} &= 2(-t_1 - t_2)\,\sigma_{w_0}
\end{aligned}$$

$$\sigma_{ts} \cdot \sigma_{ts} = 2\,\sigma_{tsts} + 2(-t_1 - t_2)\,\sigma_{sts} + (-t_2)(t_1 - 2t_2)\,\sigma_{ts}$$

$$
\begin{aligned}
\sigma_{ts} \cdot \sigma_{st} &= 2\,\sigma_{tsts} + 2\,\sigma_{stst} + (-t_2)\,\sigma_{tst} + (t_2 - 2t_1)\,\sigma_{sts} \\
\sigma_{ts} \cdot \sigma_{sts} &= \sigma_{ststs} + [2(-t_1) + (-t_2)]\,\sigma_{tsts} + (-t_1)(t_2 - 2t_1)\,\sigma_{sts} \\
\sigma_{ts} \cdot \sigma_{tst} &= 3\sigma_{ststs} + 2\sigma_{tstst} + 2(-t_1 - t_2)\,\sigma_{stst} + 3(-t_2)\,\sigma_{tsts} + (-t_2)(t_1 - 2t_2)\,\sigma_{tst} \\
\sigma_{ts} \cdot \sigma_{tsts} &= 2(-t_1 - t_2)\,\sigma_{ststs} + 2(-t_2)(-t_1 - t_2)\,\sigma_{tsts} \\
\sigma_{ts} \cdot \sigma_{stst} &= \sigma_{w_0} + [2(-t_1) + (-t_2)]\,\sigma_{tstst} + 3(-t_1)\,\sigma_{ststs} + (-t_1)(t_2 - 2t_1)\,\sigma_{stst} \\
\sigma_{ts} \cdot \sigma_{ststs} &= 2(-t_1)(-t_1 - t_2)\,\sigma_{ststs} \\
\sigma_{ts} \cdot \sigma_{tstst} &= 2(-t_1 - t_2)\,\sigma_{w_0} + 2(-t_2)(-t_1 - t_2)\,\sigma_{tstst} \\
\sigma_{ts} \cdot \sigma_{w_0} &= 2(-t_1)(-t_1 - t_2)\,\sigma_{w_0}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{st} \cdot \sigma_{st} &= 2\,\sigma_{stst} + 2(-t_1)\,\sigma_{tst} + (t_2 - 2t_1)(t_2 - t_1)\,\sigma_{st} \\
\sigma_{st} \cdot \sigma_{sts} &= 2\,\sigma_{ststs} + \sigma_{tstst} + (t_2 - 2t_1)\,\sigma_{stst} + 2(-t_1)\,\sigma_{tsts} + (t_2 - 2t_1)(t_2 - t_1)\,\sigma_{sts} \\
\sigma_{st} \cdot \sigma_{tst} &= \sigma_{tstst} + [4(-t_1) + (-t_2)]\,\sigma_{stst} + (-t_2)(-t_1 - t_2)\,\sigma_{tst} \\
\sigma_{st} \cdot \sigma_{tsts} &= \sigma_{w_0} + (-t_1 - t_2)\,\sigma_{tstst} + [4(-t_1) + (-t_2)]\,\sigma_{ststs} + (-t_2)(-t_1 - t_2)\,\sigma_{tsts} \\
\sigma_{st} \cdot \sigma_{stst} &= 2(-t_1)\,\sigma_{tstst} + 2(-t_1)(t_2 - 2t_1)\,\sigma_{stst} \\
\sigma_{st} \cdot \sigma_{ststs} &= 2(-t_1)\,\sigma_{w_0} + 2(-t_1)(t_2 - 2t_1)\,\sigma_{ststs} \\
\sigma_{st} \cdot \sigma_{tstst} &= 2(-t_1)(-t_1 - t_2)\,\sigma_{tstst} \\
\sigma_{st} \cdot \sigma_{w_0} &= 2(-t_1)(-t_1 - t_2)\,\sigma_{w_0}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{sts} \cdot \sigma_{sts} &= 2(-t_1)\sigma_{ststs} + 2(-t_1)^2\sigma_{tsts} + (-t_1)(t_2 - 2t_1)(t_2 - t_1)\,\sigma_{sts} \\
\sigma_{sts} \cdot \sigma_{tst} &= \sigma_{w_0} + [2(-t_1) + (-t_2)]\,\sigma_{tstst} + [4(-t_1) + (-t_2)]\,\sigma_{ststs} \\
&\quad + (-t_1)(t_2 - 2t_1)\,\sigma_{stst} + (-t_2)(-t_1 - t_2)\,\sigma_{tsts} \\
\sigma_{sts} \cdot \sigma_{tsts} &= 2(-t_1)(-t_1 - t_2)\,\sigma_{ststs} + (-t_1)(-t_2)(-t_1 - t_2)\,\sigma_{tsts} \\
\sigma_{sts} \cdot \sigma_{stst} &= 2(-t_1)\,\sigma_{w_0} + 2(-t_1)^2\,\sigma_{tstst} + 2(-t_1)(t_2 - 2t_1)\,\sigma_{ststs} \\
&\quad + (-t_1)(t_2 - 2t_1)(t_2 - t_1)\,\sigma_{stst} \\
\sigma_{sts} \cdot \sigma_{ststs} &= (-t_1)(-t_1 - t_2)(t_2 - 2t_1)\,\sigma_{ststs} \\
\sigma_{sts} \cdot \sigma_{tstst} &= 2(-t_1)(-t_1 - t_2)\,\sigma_{w_0} + (-t_1)(-t_2)(-t_1 - t_2)\,\sigma_{tstst} \\
\sigma_{sts} \cdot \sigma_{w_0} &= (-t_1)(-t_1 - t_2)(t_2 - 2t_1)\,\sigma_{w_0}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{tst} \cdot \sigma_{tst} &= 2(-t_1 - t_2)\,\sigma_{tstst} + 2(-t_1 - t_2)^2\,\sigma_{stst} + (-t_2)(-t_1 - t_2)(t_1 - 2t_2)\,\sigma_{tst} \\
\sigma_{tst} \cdot \sigma_{tsts} &= 2(-t_1 - t_2)\,\sigma_{w_0} + 2(-t_2)(-t_1 - t_2)\,\sigma_{tstst} + 2(-t_1 - t_2)^2\,\sigma_{ststs}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + (-t_2)(-t_1 - t_2)(t_1 - 2t_2)\,\sigma_{tsts}\\
\sigma_{tst} \cdot \sigma_{stst} &= 2(-t_1)(-t_1 - t_2)\,\sigma_{tstst} + (-t_1)(-t_1 - t_2)(t_2 - 2t_1)\,\sigma_{stst}\\
\sigma_{tst} \cdot \sigma_{ststs} &= 2(-t_1)(-t_1 - t_2)\,\sigma_{w_0} + (-t_1)(-t_1 - t_2)(t_2 - 2t_1)\,\sigma_{ststs}\\
\sigma_{tst} \cdot \sigma_{tstst} &= 3(-t_1)(-t_2)(-t_1 - t_2)\,\sigma_{tstst}\\
\sigma_{tst} \cdot \sigma_{w_0} &= 3(-t_1)(-t_2)(-t_1 - t_2)\,\sigma_{w_0}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{tsts} \cdot \sigma_{tsts} &= 3(-t_1)(-t_2)(-t_1 - t_2)\,\sigma_{ststs} + (-t_1)(-t_2)(-t_1 - t_2)(t_1 - 2t_2)\,\sigma_{tsts}\\
\sigma_{tsts} \cdot \sigma_{stst} &= 2(-t_1)(-t_1 - t_2)\,\sigma_{w_0} + (-t_1)(-t_2)(-t_1 - t_2)\,\sigma_{tstst}\\
&\quad + (-t_1)(-t_1 - t_2)(t_2 - 2t_1)\,\sigma_{ststs}\\
\sigma_{tsts} \cdot \sigma_{ststs} &= (-t_1)(-t_2)(-t_1 - t_2)(t_2 - 2t_1)\,\sigma_{ststs}\\
\sigma_{tsts} \cdot \sigma_{tstst} &= 3(-t_1)(-t_2)(-t_1 - t_2)\,\sigma_{w_0} + (-t_1)(-t_2)(-t_1 - t_2)(t_1 - 2t_2)\,\sigma_{tstst}\\
\sigma_{tsts} \cdot \sigma_{w_0} &= (-t_1)(-t_2)(-t_1 - t_2)(t_2 - 2t_1)\,\sigma_{w_0}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{stst} \cdot \sigma_{stst} &= (-t_1)(-t_1 - t_2)(t_2 - 2t_1)\,\sigma_{tstst} + (-t_1)(-t_1 - t_2)(t_2 - t_1)(t_2 - 2t_1)\,\sigma_{stst}\\
\sigma_{stst} \cdot \sigma_{ststs} &= (-t_1)(-t_1 - t_2)(t_2 - 2t_1)\,\sigma_{w_0} + (-t_1)(t_2 - t_1)(-t_1 - t_2)(t_2 - 2t_1)\,\sigma_{ststs}\\
\sigma_{stst} \cdot \sigma_{tstst} &= (-t_1)(-t_2)(-t_1 - t_2)(t_2 - 2t_1)\,\sigma_{tstst}\\
\sigma_{stst} \cdot \sigma_{w_0} &= (-t_1)(-t_2)(-t_1 - t_2)(t_2 - 2t_1)\,\sigma_{w_0}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{ststs} \cdot \sigma_{ststs} &= (-t_2)(-t_1)(t_2 - t_1)(-t_1 - t_2)(t_2 - 2t_1)\,\sigma_{ststs}\\
\sigma_{ststs} \cdot \sigma_{tstst} &= (-t_1)(-t_2)(-t_1 - t_2)(t_2 - 2t_1)\,\sigma_{w_0}\\
\sigma_{ststs} \cdot \sigma_{w_0} &= (-t_1)(-t_2)(t_2 - t_1)(-t_1 - t_2)(t_2 - 2t_1)\,\sigma_{w_0}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{tstst} \cdot \sigma_{tstst} &= (-t_1)(-t_2)(-t_1 - t_2)(t_1 - 2t_2)(t_2 - 2t_1)\,\sigma_{tstst}\\
\sigma_{tstst} \cdot \sigma_{w_0} &= (-t_1)(-t_2)(-t_1 - t_2)(t_1 - 2t_2)(t_2 - 2t_1)\,\sigma_{w_0}
\end{aligned}
$$

$$
\sigma_{w_0} \cdot \sigma_{w_0} = (-t_1)(-t_2)(t_2 - t_1)(-t_1 - t_2)(t_1 - 2t_2)(t_2 - 2t_1)\,\sigma_{w_0}
$$

APPENDIX E

Integral Chow rings of quadric bundles

In this appendix, we consider schemes over an arbitrary field k, and use the language of Chow rings rather than cohomology. We prove the following fact about odd-rank quadric bundles:

THEOREM E.1. *Let V be a vector bundle of rank $2n+1$ on a scheme X, and suppose V is equipped with a nondegenerate quadratic form. Assume there is a maximal (rank n) isotropic subbundle $F \subset V$. Let $\mathcal{Q} \xrightarrow{p} X$ be the quadric bundle of isotropic lines in V, let $h \in A^*\mathcal{Q}$ be the hyperplane class (restricted from $H = c_1(\mathcal{O}(1)) \in A^*\mathbb{P}(V)$), and let $f = [\mathbb{P}(F)] \in A^*\mathcal{Q}$. Then*

$$A^*\mathcal{Q} = A^*X[h,f]/I,$$

where the ideal I is generated by the two relations

$$\text{(E.0.1)} \qquad 2f = h^n - c_1(F)\,h^{n-1} + \cdots + (-1)^n c_n(F),$$

$$\text{(E.0.2)} \qquad f^2 = (c_n(V/F) + c_{n-2}(V/F)\,h^2 + \cdots)\,f.$$

(Here h and f have degrees 1 and n, respectively.)

A similar presentation for even-rank quadrics was first given by Edidin and Graham [**Ed-Gr1**, Theorem 7]; in fact, the second of the two relations is the same as theirs. Our purpose here is to correct a small error in the statement of the second half of their theorem (which concerned odd-rank quadrics).

Before giving the proof, we recall two basic formulas for Chern classes. Let L be a line bundle. For a vector bundle E of rank n, we have (cf. [**Fu4**, Ex. 3.2.2])

$$\text{(E.0.3)} \qquad c_n(E \otimes L) = \sum_{i=0}^{n} c_i(E)\,c_1(L)^{n-i}.$$

Also, if

$$0 \to L \to E \to E' \to 0$$

is an exact sequence of vector bundles, then inverting the Whitney formula gives

$$\text{(E.0.4)} \qquad c_k(E') = c_k(E) - c_{k-1}(E)\,c_1(L) + \cdots + (-1)^k c_1(L)^k.$$

PROOF. The classes $h, h^2, \ldots, h^{n-1}, f, f\,h, \ldots, f\,h^{n-1}$ form a basis of $A^*\mathcal{Q}$ as an A^*X-module, since they form a basis when restricted to a fiber. It is easy to see that these elements also form a basis of the ring $A^*X[h,f]/I$. Therefore it suffices to establish that the relations generating I hold in $A^*\mathcal{Q}$.

Let $i : \mathcal{Q} \hookrightarrow \mathbb{P}(V)$ be the inclusion of the quadric in the projective bundle. By [**Fu4**, Ex. 3.2.17], we have

$$i_*f = [\mathbb{P}(F)] = \sum_{i=0}^{n+1} c_i\, H^{n+1-i}$$

in $A^*\mathbb{P}(V)$, where $c_i = c_i(V/F)$. (Following the common abuse of notation, we have written c_i for p^*c_i.) On the other hand, $\mathcal{Q} \subset \mathbb{P}(V)$ is cut out by a section of $\mathcal{O}_{\mathbb{P}(V)}(2)$, so $[\mathcal{Q}] = 2\,H$ in $A^*\mathbb{P}(V)$. Therefore $i^*i_*f = 2\,h\,f$, and we have

$$2\,h\,f \quad = \quad h^{n+1} + c_1\,h^n + \cdots + c_{n+1}. \tag{E.0.5}$$

(Up to this point, we are repeating the argument of [**Ed-Gr1**].)

To prove the first relation, expand h^n in the given basis:

$$h^n = a_0\,f + a_1\,h^{n-1} + \cdots + a_n, \tag{E.0.6}$$

with $a_k \in A^kX$. Our goal is to show $a_0 = 2$, and $a_k = (-1)^{k+1}c_k(F)$ for $k > 0$.

That $a_0 = 2$ can be seen by restricting to a fiber: the Chow ring of an odd-dimensional quadric in projective space is given by $\mathbb{Z}[h, f]/(h^n - 2f, f^2)$.

Multiplying (E.0.6) by h and expanding in the basis, we have

$$h^{n+1} = 2\,h\,f + 2\,a_1\,f + (a_2 + a_1^2)\,h^{n-1} + \cdots + (a_n + a_1\,a_{n-1})\,h + a_1\,a_n.$$

On the other hand, if we rearrange and expand (E.0.5), we obtain

$$h^{n+1} = 2\,h\,f - 2\,c_1\,f - (c_2 + c_1\,a_1)h^{n-1} - \cdots - (c_n + c_1\,a_{n-1})\,h - (c_{n+1} + c_1\,a_n).$$

Comparing coefficients, we have

$$\begin{aligned} 2\,a_1 &= -2\,c_1; \\ a_k &= -c_k - a_{k-1}(a_1 + c_1) \quad (2 \le k \le n); \\ a_1\,a_n &= -c_{n+1} - c_1\,a_1. \end{aligned}$$

From the first of these equations, we see

$$a_1 + c_1 = \tau,$$

for some $\tau \in A^1X$ such that $2\,\tau = 0$. (Note that $\tau = 0$ only if $c_{n+1}(V/F) = 0$, which need not be true in general.) The remaining equations give

$$a_k \quad = \quad -c_k + c_{k-1}\,\tau - c_{k-2}\,\tau^2 + \cdots - (-1)^k\tau^k \quad (1 \le k \le n), \tag{E.0.7}$$

and $-c_{n+1} = a_n\,\tau$. (Of course, the signs on powers of τ make no difference, but we will include them as a visual aid.)

We claim $\tau = c_1(F^\perp/F)$. This can be proved in the universal case. Specifying the maximal isotropic subbundle $F \subset V$ reduces the structure group from $O(2n+1)$ to a parabolic subgroup whose Levi factor is $GL_n \times \mathbb{Z}/2\mathbb{Z}$, so the universal base is (an affine bundle over) $BGL_n \times B\mathbb{Z}/2\mathbb{Z}$. Now $A^*(BGL_n \times B\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_n, t]/(2t)$, so there is only one 2-torsion class of degree 1, namely t. Since t pulls back to $c_1(F^\perp/F)$, the claim is proved. (See [**To**] for the meaning and computation of this Chow ring. To ensure V is pulled back from Totaro's algebraic model for BG, one may have to replace X by an affine bundle or Chow envelope, as in [**Gr1**, p. 486].)

Using the exact sequence $0 \to F^\perp/F \to V/F \to V/F^\perp \to 0$ and Formula (E.0.4), Equation (E.0.7) implies

$$a_k = -c_k(V/F^\perp).$$

Since $V/F^\perp \cong F^\vee$, we obtain $a_k = (-1)^{k+1}c_k(F)$, as desired.

The second relation is proved by the argument given in [**Ed-Gr1**]. Let $j : \mathbb{P}(F) \hookrightarrow \mathcal{Q}$ be the inclusion, and let $N_{\mathbb{P}(F)/\mathcal{Q}}$ be the normal bundle. By the self-intersection formula, $j_* c_n(N_{\mathbb{P}(F)/\mathcal{Q}}) = f^2$. On the other hand, using $N_{\mathcal{Q}/\mathbb{P}(V)} = \mathcal{O}(2)$ and $N_{\mathbb{P}(F)/\mathbb{P}(V)} = V/F \otimes \mathcal{O}(1)$, and tensoring with $\mathcal{O}(-1)$, we have

$$0 \to \mathcal{O}(1) \to V/F \to N_{\mathbb{P}(F)/\mathcal{Q}} \otimes \mathcal{O}(-1) \to 0$$

on $\mathbb{P}(F)$; thus $N_{\mathbb{P}(F)/\mathcal{Q}} = ((V/F)/\mathcal{O}(1)) \otimes \mathcal{O}(1)$. By Formulas (E.0.3) and (E.0.4), we have

$$c_n(N_{\mathbb{P}(F)/\mathcal{Q}}) = c_n(V/F) + c_{n-2}(V/F)\, h^2 + \cdots .$$

The relation (E.0.2) follows after applying j_*. $\square$

Bibliography

[Ag] I. Agricola, "Old and new on the exceptional group G_2," *Notices Amer. Math. Soc.* **55** (2008), no. 8, 922–929.

[As] M. Aschbacher, "Chevalley groups of type G_2 as the group of a trilinear form," *J. Algebra* **109** (1987), 193–259.

[Ba] J. Baez, "The octonions," *Bull. Amer. Math. Soc.* **39** (2002), 145–205.

[BGG] I. Bernstein, I. Gelfand, and S. Gelfand, "Schubert cells and cohomology of the spaces G/P," *Russian Math. Surveys* **28** (1973), 1–26.

[Bi-Ha] S. Billey and M. Haiman, "Schubert polynomials for the classical groups," *J. Amer. Math. Soc.* **8**, no. 2 (1995), 443–482.

[Bi-La] S. Billey and V. Lakshmibai, *Singular loci of Schubert varieties*, Birkhäuser, 2000.

[Bo] R. Bott, "Homogeneous vector bundles," *Ann. of Math.* **66** (1957), 203–248.

[Bo-Sa] R. Bott and H. Samelson, "The integral cohomology ring of G/T," *Proc. Nat. Acad. Sci.* **7** (1955), 490–493.

[Br] R. L. Bryant, "Metrics with exceptional holonomy," *Ann. of Math.* **126** (1987), no. 3, 525–576.

[Bü-He] H. Bürgstein and W. H. Hesselink, "Algorithmic orbit classification for some Borel group actions," *Compositio Math.* **61** (1987), no. 1, 3–41.

[Ca] É. Cartan, "Le principe de dualité et la théorie des groupes simple et semi-simples," *Bull. Sci. Math.* **49** (1925), 361–374.

[Co-He] A. Cohen and A. Helminck, "Trilinear alternating forms on a vector space of dimension 7," *Comm. Algebra* **16** (1988), no. 1, 1–25.

[Co] H. S. M. Coxeter, "Integral Cayley numbers," *Duke Math. J.* **13** (1946), 561–578.

[De] M. Demazure, "Désingularisation des variétés de Schubert généralisées," *Ann. Sci. cole Norm. Sup.* **7** (1974), 53–88.

[Ed-Gr1] D. Edidin and W. Graham, "Characteristic classes and quadric bundles," *Duke Math. J.* **78** (1995), 277–299.

[Ed-Gr2] D. Edidin and W. Graham, "Equivariant intersection theory," *Invent. Math.* **131** (1998), 595–634.

[Fe-Né-Ri] L. Fehér, A. Némethi, and R. Rimányi, "Degeneracy of 2-forms and 3-forms," *Canad. Math. Bull.* **48** (2005), no. 4, 547–560.

[Fe-Ri1] L. Fehér and R. Rimányi, "Schur and Scubert polynomials as Thom polynomials—cohomology of moduli spaces," *Cent. European J. Math.* **4** (2003), 418–434.

[Fe-Ri2] L. Fehér and R. Rimányi, "Calculation of Thom polynomials for group actions," `arXiv:math/0009085v2 [math.AG]`.

[Fo-Ki] S. Fomin and A. N. Kirillov, "Combinatorial B_n-analogues of Schubert polynomials," *Trans. Amer. Math. Soc.* **348** (1996), no. 9, 3591–3620.

[Fu1] W. Fulton, "Flags, Schubert polynomials, degeneracy loci, and determinantal formulas," *Duke Math. J.* **65** (1992), no. 3, 381–420.

[Fu2] W. Fulton, "Schubert varieties in flag bundles for the classical groups," in *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry*, 241–262, Israel Math. Conf. Proc., **9**, 1996.

[Fu3] W. Fulton, "Determinantal formulas for orthogonal and symplectic degeneracy loci," *J. Differential Geom.* **43** (1996), no. 2, 276–290.

[Fu4] W. Fulton, *Intersection Theory*, second edition, Springer, 1998.

[Fu5] W. Fulton, *Equivariant cohomology in algebraic geometry*, lectures at Columbia University, notes by D. Anderson, 2007. Available at `www.math.lsa.umich.edu/~dandersn/eilenberg`.

[Fu-Ha] W. Fulton and J. Harris, *Representation Theory: A First Course*, Springer-Verlag, 1991.

[Fu-Pr] W. Fulton and P. Pragacz, *Schubert Varieties and Degeneracy Loci*, Springer, 1998.

[Ga1] S. Garibaldi, "Twisted flag varieties of trialitarian groups," *Comm. Algebra* **27** (1999), no. 2, 841–856.

[Ga2] S. Garibaldi, "Structurable algebras and groups of type E_6 and E_7," *J. Algebra* **236** no. 2 (2001), 651–691.

[Ga3] S. Garibaldi, "Cohomological invariants: exceptional groups and Spin groups," with an appendix by D. W. Hoffmann, to appear in *Mem. Amer. Math. Soc.*

[Gr1] W. Graham, "The class of the diagonal in flag bundles," *J. Differential Geometry* **45** (1997), 471–487.

[Gr2] W. Graham, "Positivity in equivariant Schubert calculus," *Duke Math. J.* **109** (2001), 599–614.

[Gri-Ram] S. Griffeth and A. Ram, "Affine Hecke algebras and the Schubert calculus," *European J. of Combin.* **25** (2004), no. 8, 1263–1283.

[Ha-Tu] J. Harris and L. Tu, "On symmetric and skew-symmetric determinantal varieties," *Topology* **23** (1984) No. 1, 71–84.

[Ha] F. R. Harvey, *Spinors and Calibrations*, Academic Press, Inc., Boston, 1990.

[Hi-Rö] L. Hille and G. Röhrle, "A classification of parabolic subgroups of classical groups with a finite number of orbits on the unipotent radical," *Transform. Groups* **4** (1999), no. 1, 35–52.

[Hu1] J. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer, 1972.

[Hu2] J. Humphreys, *Linear Algebraic Groups*, Springer, 1981.

[Il-Ma] A. Iliev and L. Manivel, "The Chow ring of the Cayley plane," Compos. Math. **141** (2005), no. 1, 146–160.

[Jó-La-Pr] T. Józefiak, A. Lascoux, and P. Pragacz, "Classes of determinantal varieties associated with symmetric and skew-symmetric matrices," *Izv. Akad. Nauk SSSR Ser. Mat.* **45** (1981), no. 3, 662–673.

[Jü-Rö] U. Jürgens and G. Röhrle, "MOP—algorithmic modality analysis for parabolic group actions," *Experiment. Math.* **11** (2002), no. 1, 57–67.

[Ke-La] G. Kempf and D. Laksov, "The determinantal formula of Schubert calculus," *Acta Math.* **132** (1974), 153–162.

[KMRT] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, American Mathematical Society, 1998.

[Kn-Mi] A. Knutson and E. Miller, "Gröbner geometry of Schubert polynomials," *Ann. Math.* **161** (2005), no. 3, 1245–1318.

[Kr-Ta] A. Kresch and H. Tamvakis, "Double Schubert polynomials and degeneracy loci for the classical groups," *Ann. Inst. Fourier (Grenoble)* **52** (2002), no. 6, 1681–1727.

[La-Ma] J. M. Landsberg and L. Manivel, "The sextonions and $E_{7\frac{1}{2}}$," *Adv. Math.* **201** (2006), no. 1, 143–179.

[La] S. Lang, *Algebra*, rev. 3rd ed., Springer, 2002.

[La-Pr] A. Lascoux and P. Pragacz, "Operator calculus for $\widetilde{Q}$-polynomials and Schubert polynomials," *Adv. in Math.* **140** (1998), 1–43.

[Ma] I. G. Macdonald, *Notes on Schubert Polynomials*, Publ. LACIM 6, Univ. de Québec à Montréal, Montréal, 1991.

[Mu] S. Mukai, "Curves, $K3$ surfaces and Fano 3-folds of genus ≤ 10," *Algebraic geometry and commutative algebra*, Vol. I, 357–377, Tokyo, 1988.

[Pa-Pr] A. Parusinski and P. Pragacz, "Chern-Schwartz-MacPherson classes and the Euler characteristic of degeneracy loci and special divisors," *J. Amer. Math. Soc.* **8** (1995), 793–817.

[Pe] H. P. Petersson, "Composition algebras over algebraic curves of genus zero," *Trans. Amer. Math. Soc.* **337** (1993), 473–493.

[Pu] S. Pumplün, "Albert algebras over curves of genus zero and one," `arXiv:0709.2308v1 [math.RA]`.

[Rich] R. W. Richardson, Jr., "Conjugacy classes in parabolic subgroups of semisimple algebraic groups," *Bull. London Math. Soc.* **6** (1974), 21–24.

[Sp] T. A. Springer, *Linear Algebraic Groups*, second edition, Birkhäuser, 1998.

[Sp-Ve] T. A. Springer and F. D. Veldkamp, *Octonions, Jordan Algebras, and Exceptional Groups*, Springer, 2000.

[Te] E. Tevelev, "Projectively dual varieties," *J. Math. Sci. (N. Y.)* **117** (2003), no. 6, 4585–4732.

[To] B. Totaro, "The Chow ring of a classifying space," *Proc. Sympos. Pure Math.* **67** (1999), AMS, 249–281.

[vdB-Sp] F. van der Blij and T. A. Springer, "Octaves and triality," *Nieuw Arch. Wisk.* **8** (1960) 158–169.