MATH 6112: ALGEBRA II
HOMEWORK #1

Due: January 18, 2019

1. Let P be a partially ordered set. (See Jacobson, p. 13, where it is called a "pre-ordered set".) The *join* of two elements x and y is the (unique) element $x \vee y$ such that
   (i) $x \leq x \vee y$,
   (ii) $y \leq x \vee y$, and
   (iii) if z is any element satisfying (i) and (ii), then $x \vee y \leq z$.
   That is, $x \vee y$ is the unique minimal element that is greater than or equal to both x and y. (Model example: P is the Boolean poset of all subsets of some fixed set, ordered by inclusion; join is set union.)
   (a) Give an example of a finite poset where joins do not exist.
   (b) Let $\boldsymbol{P}$ be the category associated to P defined in class. Show that $x \vee y$ is a coproduct in $\boldsymbol{P}$ (if it exists).
   (c) Give an example of a category in which all finite coproducts exist, but some infinite coproducts do not exist.

2. Show that initial and terminal objects in a category are unique up to unique isomorphism (if they exist).

3. Let $\mathcal{C}$ be a category with finite products and a terminal object. Recall (from class) that a *group object* is the data $(G, \cdot, \epsilon, \iota)$, where
   (a) G is an object of $\mathcal{C}$,
   and the morphisms
   (b) $G \times G \xrightarrow{\cdot} G$,
   (c) $\star \xrightarrow{\epsilon} G$,
   (d) $G \xrightarrow{\iota} G$
   satisfy appropriate commutative diagrams expressing associativity, identity, and inverse properties.
   If A is any object in $\mathcal{C}$, show that a group object G endows the set $\mathrm{Hom}(A, G)$ with the structure of a group.

4. Write down the axioms for a *ring object* R in a category $\mathcal{C}$, preferably in terms of commutative diagrams.