MATH 6112: ALGEBRA II
HOMEWORK #10

Due: April 15, 2019

1. Show that $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$, where ζ_n is a primitive nth root of unity. But show that $\mathrm{Gal}(F(\zeta_n)/F)$ can be any subgroup of $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$, for appropriate choice of F. Give an example where $\{e\} \subsetneq \mathrm{Gal}(F(\zeta_n)/F) \subsetneq \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$.

2. Let k be a field of characteristic $p > 0$, and let $E = k(t, u)$ be the field of rational functions in two (independent) variables. Consider the subfield $F = k(t^p, u^p)$. Show that E/F is finite, of degree p^2, but there are infinitely many intermediate fields $F \subset K \subset E$.

3. Let E/F be a finite extension of degree $[E : F] = n$. Recall that the **trace** map is defined by
$$\mathrm{tr}_{E/F}(\alpha) = \mathrm{tr}(m_\alpha),$$
where $m_\alpha \colon E \to E$ is the F-linear map given by multiplication by α. Show that $\mathrm{tr}_{E/F}$ is the unique map satisfying:
   (i) $\mathrm{tr}\colon E \to F$ is a homomorphism of additive groups.
   (ii) If $\alpha \in F$, then $\mathrm{tr}(\alpha) = n\alpha$.
   (iii) If $E = F(\alpha)$, and α has minimal polynomial $f = x^n + c_1 x^{n-1} + \cdots + c_n$, then $\mathrm{tr}(\alpha) = -c_1$.
   (iv) If $E \supseteq K \supseteq F$ is an intermediate field, then $\mathrm{tr}_{E/F} = \mathrm{tr}_{K/F} \circ \mathrm{tr}_{E/K}$.

   Deduce that if E/F is Galois, with group G, then
$$\mathrm{tr}_{E/F}(\alpha) = \sum_{\sigma \in G} \sigma(\alpha).$$

4. Prove the additive version of Hilbert 90: Let E/F be a cyclic extension of degree n, so $G = \mathrm{Gal}(E/F) \cong C_n$, and let $\sigma \in G$ be a generator. Then for any $\alpha \in E$, $\mathrm{tr}_{E/F}(\alpha) = 0$ if and only if there is $\beta \in E$ such that $\alpha = \beta - \sigma(\beta)$.