MATH 6112: ALGEBRA II
HOMEWORK #2

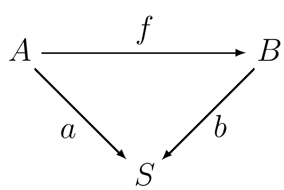
Due: January 25, 2019

1. What is a group object in the category (**Grp**) of groups?

2. (a) Suppose categories $\mathcal{C}$ and $\mathcal{D}$ have finite products and terminal objects. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor which *preserves finite products* (so $F(A \times B) \cong F(A) \times F(B)$, compatibly with the projection morphisms) and *preserves terminal objects* (so $F(\star)$ is terminal in $\mathcal{D}$ if $\star$ is terminal in $\mathcal{C}$). Show that F preserves group objects, that is, if G is a group object in $\mathcal{C}$, then $F(G)$ is naturally a group object in $\mathcal{D}$.

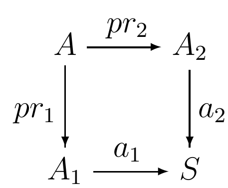
   (b) Show that the fundamental group of a topological group is abelian. (You may assume the facts that $\pi_1(X \times Y, x \times y) = \pi_1(X, x) \times \pi_1(Y, y)$ for any pointed topological spaces X and Y.)

3. Given a category $\mathcal{C}$ and an object S, the *slice category* $\mathcal{C}_S$ is the "category of objects over S": an object of $\mathcal{C}_S$ is a morphism $A \xrightarrow{a} S$ (in $\mathcal{C}$), and a morphism in $\mathcal{C}_S$ is a commuting triangle



   (of morphisms in $\mathcal{C}$). There is an evident functor $\mathcal{C}_S \to \mathcal{C}$, forgetting the morphism to S.

   On the other hand, given objects A_1 and A_2 of $\mathcal{C}$ and morphisms $a_i\colon A_i \to S$, a *fiber product* or *pullback* of A_1 and A_2 over S is an object A, equipped with morphisms $pr_i\colon A \to A_i$, making the diagram



commute, and universal with respect to this property. (If B is any other object with morphisms $f_i\colon B \to A_i$, commuting with the projections to S, then there is a unique morphism $f\colon B \to A$ so that $f_i = pr_i \circ f$.) As with products, a fiber product is unique up to unique isomorphism (if one exists).

Show that a fiber product of $a_i\colon A_i \to S$ $(i = 1, 2)$ in $\mathcal{C}$ is the same as a product in $\mathcal{C}_S$. (And interpret what "the same" means in this context.)

4. Let R_1, R_2, and S be rings, with ring homomorphisms $\varphi_i\colon R_i \to S$. Show that the fiber product of R_1 and R_2 over S exists in (**Ring**), the category of rings. (Give an explicit construction and verify the universal property.)

5. Let G and H be groups, with associated categories **G** and **H**. (Recall that these categories each have one object, and have morphisms in bijection with group elements.) Show that a functor $F\colon \mathbf{G} \to \mathbf{H}$ is the same as a group homomorphism $\varphi\colon G \to H$. Given two functors $F_1, F_2\colon \mathbf{G} \to \mathbf{H}$, show that a natural transformation $\eta\colon F_1 \Rightarrow F_2$ exists if and only if the corresponding homomorphiams φ_1, φ_2 are conjugate. (I.e., there exists $h \in H$ such that $\varphi_2(g) = h \cdot \varphi_1(g) \cdot h^{-1}$ for all $g \in G$.)