MATH 6112: ALGEBRA II
HOMEWORK #3

Due: February 1, 2017

1. Consider the category (**CRing**) of commutative rings (with unit). Fix a ring A, with a multiplicative set $S \subseteq A$ and an ideal $I \subseteq A$. Define a functor $F_{A,S,I} \colon (\mathbf{CRing}) \to (\mathbf{Set})$ by

$$F_{A,S,I}(B) = \left\{ A \xrightarrow{g} B \mid g(s) \text{ is a unit for all } s \in S \text{ and } g(x) = 0 \text{ for all } x \in I \right\},$$

   with functoriality coming from the usual composition. (That is, regarding $F_{A,S,I}(B)$ as a subset of $\mathrm{Hom}(A, B)$, for a homomorphism $B \xrightarrow{f} B'$, the homomorphism $F_{A,S,I}(B) \to F_{A,S,I}(B')$ is induced from $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B')$.)

   Show that $F_{A,S,I}$ is representable. (You may want to look at the "trivial" cases: $S = \{1\}$ or $I = (0)$.) Use the Yoneda embedding to conclude that the operations of localization by a multiplicative set and taking the quotient by an ideal commute.

2. In the category (**Hdf**) of Hausdorff topological spaces (with continuous maps), give an example of a morphism which is epi but not surjective. But show that every monic in (**Top**) is injective.

3. Prove the dual Yoneda Lemma: if F is a *contravariant* functor from $\mathcal{C}$ to (**Set**), then for any object X of $\mathcal{C}$, there is a (natural) bijection between $F(X)$ and the set of natural transformations from $h_X = \mathrm{Hom}_{\mathcal{C}}(\cdot, X)$ to F.

   (Then convince yourself that you have described a fully faithful embedding functor $h_{(\cdot)} \colon \mathcal{C} \to (\mathbf{Set})^{\mathcal{C}^{\mathrm{op}}}$.)

4. Let $k[x]$ be the polynomial ring over a field, and let $A_i = k[x]/(x^{i+1})$ be the ring of truncated polynomials, for integers $i \geq 0$. For $i \leq j$, there is a trunction homomorphism $A_j \to A_i$. Show that the ring of formal power series,

$$k[[x]] = \{a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n \geq 0} a_n x^n \mid a_n \in k\},$$

   with usual addition and multiplication, equipped with projection homomorphisms $p_i \colon k[[x]] \to A_i = k[x]/(x^{i+1})$ defined by truncation, is the (inverse) limit of the directed system A_i in the category of rings.