MATH 6112: ALGEBRA II
HOMEWORK #4

Due: February 8, 2019

1. Let R be a ring. In the category $(R - \mathbf{mod})$ of (left) R-modules, show that every monic is the kernel of its cokernel, and every epi is the cokernel of its kernel. (Use the categorical notions of *kernel* and *cokernel* as defined in class.)

2. In the category $(R-\mathbf{mod})$, show that the (covariant and contravariant) Hom functors $\mathrm{Hom}_R(\cdot, M)$ and $\mathrm{Hom}_R(M, \cdot)$ are left exact.

3. Give examples of modules M showing that the functors $\mathrm{Hom}_R(\cdot, M)$ and $\mathrm{Hom}_R(M, \cdot)$ are not necessarily exact. (One example for each functor is enough.)

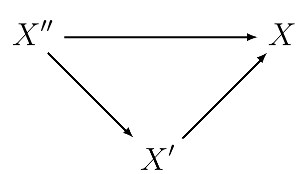
4. For $m > 0$, show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (m\mathbb{Z})$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$ as $\mathbb{Z}$-modules (abelian groups). Use this to construct an example showing that the functors $M \otimes_R (\cdot)$ and $(\cdot) \otimes_R M$ need not be left-exact.

5. Given a functor $F\colon \mathcal{C} \to \mathcal{S}$, say an object X of $\mathcal{C}$ lies "over" an object T of $\mathcal{S}$ if $F(X) = T$; similarly, a morphism $p\colon X' \to X$ lies over a morphism $f\colon T' \to T$ if $F(p) = f$.

   A **category fibered in groupoids (CFG)** is a functor $\mathcal{C} \to \mathcal{S}$, which satisfies the following two axioms:

   (i) Given a morphism $T' \xrightarrow{f} T$ in $\mathcal{S}$ and an object X of $\mathcal{C}$ over T, there exists an object X' over T' together with a morphism $X' \xrightarrow{p} X$ in $\mathcal{C}$ over f.

   (ii) Given $T'' \to T' \to T$ in $\mathcal{S}$, with $X' \to X$ in $\mathcal{C}$ over $T' \to T$ and $X'' \to X$ over $T'' \to T$, there exists a unique morphism $X'' \to X'$ over $T'' \to T'$ so that

   commutes.

Roughly speaking, these axioms say "fiber products (i) exist and (ii) are unique".

Now let $\mathcal{S} = (\mathbf{Top})$ be the category of topological spaces with continuous maps. Consider the category $(\mathbf{Tri})$ defined as follows.

- An object is a continuous map $\pi\colon X \to T$, with a metric on fibers, $X \times_T X \to \mathbb{R}$, such that each fiber $X_t = \pi^{-1}(t)$ is isometric to a plane triangle in $\mathbb{R}^2$.
- A morphism $(X' \to T') \to (X \to T)$ is commuting square of continuous maps

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ T' & \longrightarrow & T \end{array}$$

  such that $X' \cong X \times_T T'$ (i.e., X' is homeomorphic to the fiber product).

There is an evident functor $(\mathbf{Tri}) \to \mathcal{S}$. (What is it on objects and morphisms??)

(a) Show that $(\mathbf{Tri}) \to \mathcal{S}$ is a CFG.

(b) Consider the topological space $\widetilde{T} = \{(a, b, c) \in \mathbb{R}^3 \mid a + b > c, a + c > b, b + c > a\} \subseteq \mathbb{R}^3$. Construct an object $\widetilde{X} \to \widetilde{T}$ of $(\mathbf{Tri})$ whose fiber over (a, b, c) is a triangle with these side lengths. Explain why this is *not* a terminal object in $(\mathbf{Tri})$. (Hint: imagine an equilateral triangle rotating $60°$ as it moves around a circle.)

This category $(\mathbf{Tri})$ (the "category of triangles") is a prototypical example of the notion of a (topological) **stack**.