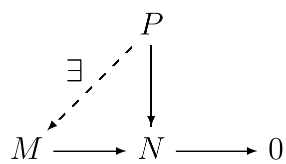
MATH 6112: ALGEBRA II
HOMEWORK #5

Due: February 18, 2019

1. Show that the following are equivalent, for an R-module P:
   (i) P satisfies the lifting property: given a surjective homomorphism $M \to N$ and a homomomorphism $P \to N$, there is a lift $P \to M$ as in the diagram below.



   (ii) The functor $\mathrm{Hom}_R(P, \cdot)$ is exact.
   (iii) P is a direct summand of a free module. That is, there is an R-module Q, a free R-module F, and an isomorphism $P \oplus Q \cong F$.

2. If $e \in R$ is idempotent (i.e., $e^2 = e$), show that Re is a projective R-module.

3. For $n > 1$, show that the $M_n(R)$-module R^n is projective, but not free. (If it helps, you may assume that R, and hence $M_n(R)$, is Noetherian, as well as the following fact about Noetherian rings A: if $A^a \cong A^b$, then $a = b$.)

4. Let
$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$
and
$$\cdots \to P_2' \xrightarrow{d_2'} P_1' \xrightarrow{d_1'} P_0' \xrightarrow{d_0'} M' \to 0$$
be projective resolutions. Show that a homomorphism $\varphi \colon M \to M'$ lifts to a homorphism $f_\bullet \colon P_\bullet \to P_\bullet'$ of complexes.

5. Let R be a commutative ring, and M an R-module.
   (a) Let $S \subseteq R$ a multiplicative set. Show that $S^{-1}R \otimes_R M \cong S^{-1}M$ (as $S^{-1}R$-modules).
   (b) Let $\mathfrak{a} \subseteq R$ an ideal. Show that $(R/\mathfrak{a}R) \otimes_R M \cong M/\mathfrak{a}M$ as R-modules (and as $R/\mathfrak{a}R$-modules).