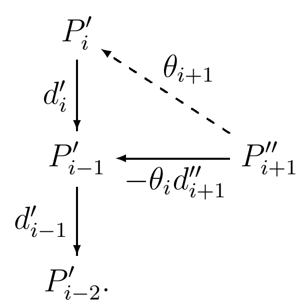
MATH 6112: ALGEBRA II
HOMEWORK #7

Due: March 4, 2019

1. Given a short exact sequence of R-modules
$$0 \to M' \to M \to M'' \to 0,$$
complete the construction of a short exact sequence of projective resolutions
$$0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$$
as follows. Recall that we have chosen resolutions $(P'_\bullet, d'_\bullet)$ and $(P''_\bullet, d''_\bullet)$ for M' and M'', and we set $P_i = P'_i \oplus P''_i$. For $i > 0$, the differentials should be defined as
$$d_i(x', x'') = (d'_i x' + \theta_i x'',\ d''_i x''),$$
for some $\theta_i \colon P''_i \to P'_{i-1}$ fitting into a diagram



(Hint: Use the fact that d'_i maps onto the kernel of d'_{i-1}, and $-\theta_i d_{i+1}$ maps into this kernel, together with the lifting property of projective modules, to construct θ_{i+1}. Note that in the construction of θ_1 in class the map d'_i in the above diagram was replaced by a map $\alpha\epsilon' \colon P_0 \to M$.)

2. Let R be a commutative PID. For a nonzero element $a \in R$, let $M = R/(a)$. For any R-module N, show that $\mathrm{Ext}^1_R(M, N) \cong N/aN$.

3. Fix the base ring $R = \mathbb{Z}$. For a group G and a G-module A, let $H^n(G, A)$ be defined via the explicit cochain complex $C^\bullet(G, A)$. Show that $H^1(G, A) \cong \mathrm{Ext}^1_{\mathbb{Z}[G]}(\mathbb{Z}, A)$. (Hint: use the exact sequence of $\mathbb{Z}[G]$-modules $0 \to K \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ to describe Ext^1, as in class.)

4. Compute $H^0(G, A)$ and $H^1(G, A)$ for the following groups and modules.
   (a) $G = \mathbb{Z}/2\mathbb{Z}$ acting trivially on $A = \mathbb{Z}/2\mathbb{Z}$. ("Trivial" means $g \cdot a = a$ for all g and all a.)
   (b) $G = \mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ acting by the sign representation on $A = \mathbb{Z}/3\mathbb{Z}$.
   (c) $G = \mathbb{Z}/n\mathbb{Z} = \{\mathrm{e}^{2\pi i k/n} \mid k = 0, \ldots, n-1\}$ acting by multiplication on $A = \mathbb{C}$ (the additive group of complex numbers).