MATH 6112: ALGEBRA II
HOMEWORK #8

Due: March 29, 2019

1. Consider a diagram of R-modules (or objects in an abelian category)
$$\begin{array}{ccccccccc} A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \\ \alpha \uparrow & & \beta \uparrow & & \gamma \uparrow & & \delta \uparrow & & \epsilon \uparrow \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E, \end{array}$$
with exact rows. View this as the 0th page of a first-quadrant spectral sequence, so $E_0^{1,1} = A$, $E_0^{2,1} = B$, $E_0^{1,2} = A'$, etc. Check that most terms of $E_\infty^{\bullet,\bullet}$ are zero, and use this to prove the **five-lemma**: asssuming that some of $\alpha, \beta, \delta, \epsilon$ are either injective or surjective, deduce that γ is injective; similarly, assuming some (different choices) of $\alpha, \beta, \delta, \epsilon$ are either injective or surjective, deduce that γ is surjective.

2. Let F be a field, and $E = F(u)$ an extension field, where u is algebraic of odd degree over F. (That is, its minimal polynomial has odd degree.) Show that $E = F(u^2)$.

3. Let E be a finite extension of F, of degree $[E : F] = n$. Let K/F be any extension. Show that the number of embeddings $E \hookrightarrow K$ which restrict to the identity on F is at most n.

4. Let k be any field, and $E = k(t)$ the field of rational functions in one variable.
   (a) Define automorphisms σ and τ of E by setting $\sigma(\varphi)(t) = \varphi(1-t)$ and $\tau(\varphi)(t) = \varphi(t^{-1})$, for $\varphi \in k(t)$. Show that σ and τ generate a group G of automorphisms which is isomorphic to S_3, the symmetric group on 3 letters.
   (b) Let $\psi = \frac{(t^2-t+1)^3}{t^2(t-1)^2}$. Show that the fixed field E^G is equal to $k(\psi)$.

5. Let $f(x) \in F[x]$ be irreducible, and E/F a (finite) normal extension. Show that the irreducible factors of f in $E[x]$ are all of the same degree, and are all conjugate over F. That is, given any two factors, there is an automorphism in $\mathrm{Gal}(E/F)$ which takes one to the other.

6. Let
$$f = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$
be a polynomial in $F[x]$. Its **formal derivative** is the polynomial
$$f' = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$
Now assume f is monic and has positive degree. Show that f is separable if and only if the gcd (f, f') equals 1. If f is also irreducible, show that it is separable if and only if $f' \neq 0$. Deduce that every field of characteristic 0 is **perfect** (i.e., all irreducible polynomials are separable).