MATH 6112: ALGEBRA II
HOMEWORK #9

Due: April 5, 2019

1. Let F be a field of characteristic $p > 0$. Show that F is perfect iff $F = F^p$, as follows.

   (a) Check that $F^p := \{\alpha^p \mid \alpha \in F\}$ is a subfield. If $a \in F$ but $a \notin F^p$, show that $x^p - a$ is irreducible but not separable, so F is not perfect.

   (b) If F is not perfect and $f \in F[x]$ is irreducible and not separable, show that some coefficient of f must not be a pth power, so $F \neq F^p$.

   Deduce that all finite fields are perfect.

2. Let E/F be a finite Galois extension, with Galois group G. Suppose $E = K \cdot L$ for two intermediate extensions K, L such that $K \cap L = F$, and L/F is a normal extension. Let $H = \text{Gal}(E/K)$ and $N = \text{Gal}(E/L)$. Show that N is a normal subgroup of G, that $G = H \cdot N$, and that $H \cap N = \{e\}$. Deduce that G is a semidirect product $N \rtimes H$, and that G is a direct product if and only if K/F is also a normal extension.

3. Let $F = \mathbb{F}_q$ be a finite field with $q = p^n$ elements. Let E/F be a finite extension of degree m. Show that E/F is Galois, with cyclic Galois group, as follows.

   (a) Let $\mathcal{F}\colon E \to E$ be the automorphism defined by $\mathcal{F}(\alpha) = \alpha^p$. Let $\mathcal{F}_q = \mathcal{F}^n$, so $\mathcal{F}_q(\alpha) = \alpha^q$. Show that $\mathcal{F}_q \in \text{Gal}(E/F)$.

   (b) If $[E : F] = m$, show that the order of $\mathcal{F}_q$ is at most m.

   (c) Show that the order of $\mathcal{F}_q$ equals m.

   (d) Let $F' = E^{\langle \mathcal{F}_q \rangle}$, and show that $F = F'$. Conclude that E/F is Galois, with group $\text{Gal}(E/F)$ generated by $\mathcal{F}_q$.

   The automorphism $\mathcal{F}$ is called the *Frobenius automorphism*, and exists for any field of characteristic $p > 0$.

4. Let E/F be the splitting field of a polynomial $f \in F[x]$. Show that $\text{Gal}(E/F)$ acts transitively on the roots of f if and only if f is a power of an irreducible polynomial (over F).