HOMEWORK #1

Due: September 11, 2026

1. Let $\{\mathfrak{p}_i\}$ be a nonempty family of prime ideals, and suppose that the $\mathfrak{p}_i$ are totally ordered by inclusion. Show that $\bigcap \mathfrak{p}_i$ is a prime ideal. Deduce that for any proper ideal I in a ring A, the set of prime ideals containing I has a minimal element.

2. Let $(A, \mathfrak{m})$ be a local ring such that $\mathfrak{m}$ is principal and $\bigcap \mathfrak{m}^n = (0)$. Show that A is noetherian and every nonzero ideal is a power of $\mathfrak{m}$.

3. Let $f\colon A \to B$ be a ring homomorphism, let $S \subseteq A$ be a multiplicative set, and let $T = f(S)$. Show that $S^{-1}B$ and $T^{-1}B$ are isomorphic as $S^{-1}A$-modules.

4. Let $\mathfrak{p} \subset A$ be a prime ideal, and consider the nth *symbolic power*:
$$\mathfrak{p}^{(n)} := (\mathfrak{p}^n A_{\mathfrak{p}}) \cap A.$$
Show that $\mathfrak{p}^{(n)}$ is primary, with radical $\mathfrak{p}$.

5. If $S \subset A$ is a multiplicative set, show that $\operatorname{Spec} S^{-1}A$ is homeomorphic to the subspace $\{\mathfrak{p} \mid \mathfrak{p} \cap S = \emptyset\}$ of $\operatorname{Spec} A$, via the map associated to the canonical homomorphism $A \to S^{-1}A$. Give examples showing this can be neither open nor closed in $\operatorname{Spec} A$.

   When $S = \{1, a, a^2, \ldots\}$, so $S^{-1}A = A_a$, show that $\operatorname{Spec} A_a$ is homeomorphic to the distinguished open set $D(a) \subseteq \operatorname{Spec} A$.

   Given an ideal $I \subseteq A$, show that the canonical homomorphism $A \to A/I$ induces a homeomorphism $\operatorname{Spec} A/I \xrightarrow{\sim} V(I) \subseteq \operatorname{Spec} A$.