HOMEWORK #4

Due: October 26, 2026

1. Let $R = k[x_1, x_2, \ldots]$ be a polynomial ring in countably many variables over a field. Let

$$\begin{aligned}
\mathfrak{p}_1 &= (x_1, \ldots, x_{d(1)}), \\
\mathfrak{p}_2 &= (x_{d(1)+1}, \ldots, x_{d(2)}), \\
&\vdots \\
\mathfrak{p}_m &= (x_{d(m-1)+1}, \ldots, x_{d(m)}), \\
&\vdots
\end{aligned}$$

   be an infinite collection of primes; the point is their generators come from disjoint sets of variables. Let $S = R \smallsetminus \bigcup \mathfrak{p}_m$, so S is a multiplicative set, and let $A = S^{-1}R$.

   (a) Show that the maximal ideals of A are exactly the ideals $S^{-1}\mathfrak{p}_m$, and conclude that $\dim A = \sup\{d(m) - d(m-1)\}_{m \geq 1}$. In particular, A may be infinite-dimensional.

   (b) Let B be any ring such that $B_{\mathfrak{m}}$ is noetherian for every maximal ideal, and such that every element of B is contained in only finitely many maximal ideals of B. Prove that B is noetherian.

   (c) Show that the ring $A = S^{-1}R$ defined above is noetherian.

2. Let R be a noetherian ring. Show that $\dim R[x, x^{-1}] = \dim R + 1$.

3. Let a_1, a_2 be a regular sequence in a domain A. Show that $a_1 x - a_2$ generates a prime ideal of $A[x]$.

4. Let $(A, \mathfrak{m}, k)$ be a regular local ring, with $K = \operatorname{Frac}(A)$. For a nonzero $a \in A$, define $\nu(a) = i$ if $a \in \mathfrak{m}^i$ but $a \notin \mathfrak{m}^{i+1}$. Show that ν extends to an additive valuation of K. Let R be the valuation ring of ν, that is, $R = \{x \in K^\times \mid \nu(x) \geq 0\} \cup \{0\}$. Let $x_1, \ldots, x_d$ be a regular s.o.p. of A, let $B = A[x_2/x_1, \ldots, x_d/x_1]$, and let $\mathfrak{p} = x_1 B$. Then $\mathfrak{p}$ is prime in B, and $R = B_{\mathfrak{p}}$.