HOMEWORK #1

Due: September 8, 2021

1. Which of the following are (closed) algebraic subsets of the given affine space? (Give reasons.)
   i. The set of diagonalizable $n \times n$ complex matrices, in $\mathbb{A}^{n^2}(\mathbb{C})$.
   ii. The set of nilpotent $n \times n$ matrices, in $\mathbb{A}^{n^2}(\mathbb{C})$.
   iii. The set $\{(t, \mathrm{e}^t) \mid t \in \mathbb{C}\}$, in $\mathbb{A}^2(\mathbb{C})$.
   iv. The set $\{(\sin(t), \cos(t)) \mid t \in \mathbb{R}\}$, in $\mathbb{A}^2(\mathbb{R})$.

2. For affine varieties $X, Y \subseteq \mathbb{A}^n$, show that $I(X \cap Y) = \sqrt{I(X) + I(Y)}$. Give an example where taking the radical is necessary.

3. Assume the base field k is algebraically closed. The *tangent lines* to a plane curve $\{f(x, y) = 0\}$ at the origin $(0, 0)$ are the lines determined by the irreducible (linear) factors of f_m, the lowest nonzero homogeneous part of f. Given any set of d lines through the origin, show that there exists an irreducible plane curve of degree $d + 1$ having these lines as its tangent lines.

   (Hint: Suppose $F \in k[x_1, \dots, x_n]$ is homogeneous of degree m, and G is homogeneous of degree $m+1$. If F and G have no common factors, show that $F + G$ is irreducible.)

4. Show that any algebraic set in $\mathbb{A}^n(\mathbb{R})$ can be defined by a single polynomial.

5. Find the image of the morphism $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ defined by $f(x, y) = (x, xy)$. Is it dense? closed?

6. Prove that for any morphism $f\colon X \to Y$ of affine varieties, there exists a morphism $\Gamma_f\colon X \to X \times Y$, which is an isomorphism of X onto a closed subset of $X \times Y$, such that $f = p_2 \circ \Gamma_f$. Here $p_2\colon X \times Y \to Y$ is the projection onto the second factor.

   This shows that every morphism is the composite of a closed embedding and a projection. The morphism Γ_f is called the *graph* of f. (Sometimes the image $\Gamma_f(X) \subseteq X \times Y$ is also called the graph.)

7. [Gathmann, Exercise 1.4.2]

8. [Gathmann, Exercise 1.4.3]