HOMEWORK #1

Due: September 8, 2017

1. Which of the following are (closed) algebraic subsets of the given affine space? (Give reasons.)
   i. The set of diagonalizable \( n \times n \) complex matrices, in \( \mathbb{A}^{n^2}(\mathbb{C}) \).
   ii. The set of nilpotent \( n \times n \) matrices, in \( \mathbb{A}^{n^2}(\mathbb{C}) \).
   iii. The set \( \{(t,e^t) \mid t \in \mathbb{C}\} \), in \( \mathbb{A}^2(\mathbb{C}) \).
   iv. The set \( \{(\sin(t),\cos(t)) \mid t \in \mathbb{R}\} \), in \( \mathbb{A}^2(\mathbb{R}) \).

2. For affine varieties \( X, Y \subseteq \mathbb{A}^n \), show that \( I(X \cap Y) = \sqrt{I(X) + I(Y)} \).
   Give an example where taking the radical is necessary.

3. Assume the base field \( k \) is algebraically closed. The tangent lines to a plane curve \( \{f(x,y) = 0\} \) at the origin \((0,0)\) are the lines determined by the irreducible (linear) factors of \( f_m \), the lowest nonzero homogeneous part of \( f \). Given any set of \( d \) lines through the origin, show that there exists an irreducible plane curve of degree \( d+1 \) having these lines as its tangent lines.
   (Hint: Suppose \( F \in k[x_1,\ldots,x_n] \) is homogeneous of degree \( m \), and \( G \) is homogeneous of degree \( m+1 \). If \( F \) and \( G \) have no common factors, show that \( F+G \) is irreducible.)

4. Show that any algebraic set in \( \mathbb{A}^n(\mathbb{R}) \) can be defined by a single polynomial.

5. Find the image of the morphism \( f: \mathbb{A}^2 \to \mathbb{A}^2 \) defined by \( f(x,y) = (x,xy) \). Is it dense? closed?

6. Prove that for any morphism \( f: X \to Y \) of affine varieties, there exists a morphism \( \Gamma_f: X \to X \times Y \), which is an isomorphism of \( X \) onto a closed subset of \( X \times Y \), such that \( f = p_2 \circ \Gamma_f \). Here \( p_2: X \times Y \to Y \) is the projection onto the second factor.
   This shows that every morphism is the composite of a closed embedding and a projection. The morphism \( \Gamma_f \) is called the graph of \( f \). (Sometimes the image \( \Gamma_f(X) \subseteq X \times Y \) is also called the graph.)

7. [Gathmann, Exercise 1.4.2]

8. [Gathmann, Exercise 1.4.3]