# HOMEWORK #3

Due: September 29, 2021

1. Let $\mathbb{P}M_{n,p}^{\le r} \subseteq \mathbb{P}^{np-1}$ be the set of $n \times p$ matrices of rank at most $r$, up to scalar multiple.

   (a) Identify a homogeneous ideal $I_r$ such that $\mathbb{P}M_{n,p}^{\le r} = Z(I_r)$.

   (b) Show that $\mathbb{P}M_{n,p}^{\le r}$ is irreducible. (Think about multiplication by $GL_n \times GL_p$, and exercise 3 from HW#2.)

2. The *Veronese embedding* $v_d \colon \mathbb{P}^n \hookrightarrow \mathbb{P}^N$, for $N = \binom{n+d}{d} - 1$, is defined by

   $$v_d([X_0, \dots, X_n]) = [(\text{all monomials of degree } d \text{ in the } X_i\text{'s})].$$

   Show that the image $v_d(\mathbb{P}^n) \subseteq \mathbb{P}^N$ is not contained in any hyperplane.

3. Suppose $X \subseteq \mathbb{P}^2$ is a plane conic. Show that $\mathbb{P}^2 \smallsetminus X$ is an affine variety. (Use the Veronese embedding.)

4. Let $X = Z(x - y^2, yz - 1) \subseteq \mathbb{A}^3$. Find generators for the homogeneous ideal of $\overline{X} \subseteq \mathbb{P}^3$. (View $\mathbb{A}^3$ as the standard affine open set $U_0 \subseteq \mathbb{P}^3$.)

5. For any $d \ge 0$, let $C_d \subseteq \mathbb{P}^3$ be the curve which is the image of the map

   $$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^3 \\ [s,t] &\mapsto [s^d, s^{d-1}t, st^{d-1}, t^d]. \end{aligned}$$

   Note that $C_d$ lies in the quadric surface $\{XW - YZ = 0\}$, which is identified with $\mathbb{P}^1 \times \mathbb{P}^1$ by the Segre embedding.
   (a) Find a single bi-homogeneous polynomial defining $C_d \subseteq \mathbb{P}^1 \times \mathbb{P}^1$.
   (b) Find homogeneous polynomials defining $C_d \subseteq \mathbb{P}^3$.
   (c) Show that $C_d \cong \mathbb{P}^1$.

6. Gathmann, Exercise 2.6.13.