HOMEWORK #5

Due: October 20, 2017

1. Compute the dimension of $M_{n,m}^{\leq r}$, the set of $n \times m$ matrices having rank at most $r$, for some $r \leq n,m$. (Hint: Parametrize certain subspaces of the kernel of such a matrix, considered as a linear map $k^m \to k^n$.)

2. Consider lines on a threefold $\{ F = 0 \} \subseteq \mathbb{P}^4$.
   (a) Find a (minimal) integer $N$ such that for all $d > N$, a general threefold of degree $d$ contains no lines. (“General” means: for every $F$ in some nonempty open subset of $\mathbb{P}(\text{Sym}^d k^5)$.)
   (b) For $d$ equal to the bound $N$ from (a), show that every threefold of degree $d$ contains at least one line, and general threefolds of this degree contain finitely many lines.

3. Consider an $n$-dimensional projective variety $X \subseteq \mathbb{P}^N$ (so $n \leq N$). The secant variety $\text{Sec}(X) \subseteq \mathbb{P}^N$ is the variety swept out by all lines through two points of $X$. More precisely, define a closed subset $Z_X := \{ (x, y, p) \mid x \neq y \text{ and } p \in \overline{xy} \} \subseteq X \times X \times \mathbb{P}^N$, where $\overline{xy} \subseteq \mathbb{P}^N$ is the line through points $x$ and $y$. Then $\text{Sec}(X)$ is the image of $Z_X$ under the third projection.
   (a) Show that $\text{Sec}(X)$ is irreducible, of dimension at most $2n + 1$.
   (b) For $X = \nu_2(\mathbb{P}^2) \subseteq \mathbb{P}^5$ (the Veronese surface), show that $\text{Sec}(X) \neq \mathbb{P}^5$. So the secant variety can be smaller than “expected”, even for varieties which are not contained in any linear subspace.

4. Show that the $k$-algebra $k[x, y]/(xy(x - y))$ is not isomorphic to the $k$-algebra $O(Z)$, where $Z \subseteq \mathbb{A}^3$ is the union of the three coordinate lines; also find a presentation of $O(Z)$. (Cf. [Shafarevich, §II.1, Ex. 5-6].)

5. Prove that the local ring of the curve $\{ xy = 0 \} \subseteq \mathbb{A}^2$ at $p = (0,0)$ is isomorphic to the subring $O \subseteq O_{\mathbb{A}^1,0} \oplus O_{\mathbb{A}^1,0}$ consisting of functions $f_1, f_2$ such that $f_1(0) = f_2(0)$. 
6. Let \( F(X, Y, Z) = 0 \) be the equation of an irreducible curve \( C \subseteq \mathbb{P}^2 \), over a field of characteristic zero. Consider the rational map \( \varphi: C \dashrightarrow \mathbb{P}^2 \) given by

\[
\varphi([a, b, c]) = \left[ \frac{\partial F}{\partial X}(a, b, c), \frac{\partial F}{\partial Y}(a, b, c), \frac{\partial F}{\partial Z}(a, b, c) \right].
\]

Show that \( \varphi(C) \) is a point if and only if \( C \) is a line, and that if \( C \) is not a line, then \( \varphi \) is regular at \( p \in C \) if and only if \( x \) is a nonsingular point.

The (closure of the) image \( \varphi(C) \subseteq \mathbb{P}^2 \) is called the dual curve of \( C \). Show that the dual curve of a nonsingular conic is a nonsingular conic.

What is an intrinsic description of the map \( \varphi \)?

7. [Gathmann, Exercise 4.6.6]