HOMEWORK #5

Due: October 22, 2021

1. Compute the dimension of $M_{n,m}^{\leq r}$, the set of $n \times m$ matrices having rank at most r, for some $r \leq n, m$. (Hint: Parametrize certain subspaces of the kernel of such a matrix, considered as a linear map $k^m \to k^n$.)

2. Consider lines on a threefold $\{F = 0\} \subseteq \mathbb{P}^4$.

   (a) Find a (minimal) integer N such that for all $d > N$, a general threefold of degree d contains no lines. ("General" means: for every F in some nonempty open subset of $\mathbb{P}(\mathrm{Sym}^d k^5)$.)

   (b) For d equal to the bound N from (a), show that every threefold of degree d contains at least one line. *You may assume the existence of a* **single** *threefold X_0 with whatever property you desire.*

   (c) In the situation of (b), show that general threefolds of this degree contain finitely many lines.

3. Consider an n-dimensional projective variety $X \subseteq \mathbb{P}^N$ (so $n \leq N$). The *secant variety* $\mathrm{Sec}(X) \subseteq \mathbb{P}^N$ is the variety swept out by all lines through two points of X. More precisely, define a closed subset

$$Z_X := \overline{\{(x, y, p) \mid x \neq y \text{ and } p \in \overline{xy}\}} \subseteq X \times X \times \mathbb{P}^N,$$

   where $\overline{xy} \subseteq \mathbb{P}^N$ is the line through points x and y. Then $Sec(X)$ is the image of Z_X under the third projection.

   (a) Show that $\mathrm{Sec}(X)$ is irreducible, of dimension at most $2n + 1$.

   (b) For $X = \nu_2(\mathbb{P}^2) \subseteq \mathbb{P}^5$ (the Veronese surface), show that $\mathrm{Sec}(X) \neq \mathbb{P}^5$. So the secant variety can be smaller than "expected", even for varieties which are not contained in any linear subspace.

4. Show that the k-algebra $k[x, y]/(xy(x - y))$ is not isomorphic to the k-algebra $\mathcal{O}(Z)$, where $Z \subseteq \mathbb{A}^3$ is the union of the three coordinate lines; also find a presentation of $\mathcal{O}(Z)$. (Cf. [Shafarevich, §II.1, Ex. 5-6].)

5. Prove that the local ring of the curve $\{xy = 0\} \subseteq \mathbb{A}^2$ at $p = (0,0)$ is isomorphic to the subring $\mathcal{O} \subseteq \mathcal{O}_{\mathbb{A}^1,0} \oplus \mathcal{O}_{\mathbb{A}^1,0}$ consisting of functions f_1, f_2 such that $f_1(0) = f_2(0)$.

6. Let $F(X,Y,Z) = 0$ be the equation of an irreducible curve $C \subseteq \mathbb{P}^2$, over a field of characteristic zero. Consider the rational map $\varphi\colon C \dashrightarrow \mathbb{P}^2$ given by
$$\varphi([a,b,c]) = \left[\frac{\partial F}{\partial X}(a,b,c),\ \frac{\partial F}{\partial Y}(a,b,c),\ \frac{\partial F}{\partial Z}(a,b,c)\right].$$
Show that $\varphi(C)$ is a point if and only if C is a line, and that if C is not a line, then φ is regular at $p \in C$ if and only if x is a nonsingular point.

   The (closure of the) image $\varphi(C) \subseteq \mathbb{P}^2$ is called the *dual curve* of C. Show that the dual curve of a nonsingular conic is a nonsingular conic.

   What is an intrinsic description of the map φ?

7. [Gathmann, Exercise 4.6.6]