HOMEWORK #6

Due: November 7, 2018

1. (a) Show that a general hypersurface $X = \{ F = 0 \} \subseteq \mathbb{P}^n$, given by a degree-$d$ polynomial in $n+1$ variables, is nonsingular. (Here, as usual, “general” means this is true for all $F$ in an open dense set of the projective space $\mathbb{P}^{N(d,n)}$ of all degree-$d$ polynomials in $n+1$ variables.)

(b) Show that there is an irreducible homogeneous polynomial $\Delta(d, n)$ in $\binom{n+d}{d}$ variables, unique up to nonzero scalar multiple, such a degree-$d$ hypersurface $\{ F = 0 \} \subseteq \mathbb{P}^n$ is singular iff $\Delta(d, n)$ vanishes on the coefficients of $F$.

The polynomial $\Delta$ is called the discriminant. (Computing it explicitly is a nontrivial problem!)

2. Let $S$ be a surface (dim $S = 2$) and $p \in S$ a nonsingular point. Let $C$ and $D$ be two curves in $S$ passing through $p$, both nonsingular at $p$. Let $\pi: \tilde{S} \to S$ be the blowup of $S$ at $p$, and set $\tilde{C} = \pi^{-1}(C \setminus p)$ and $\tilde{D} = \pi^{-1}(D \setminus p)$. Let $E = \pi^{-1}(p)$. Show that $C \cap E = D \cap E$ if and only if $C$ and $D$ are tangent at $p$. (This means $T_p C = T_p D$ as subspaces of $T_p S$.)

3. Consider the rational map $\varphi: \mathbb{P}^2 \to \mathbb{P}^4$ given by

$$[x, y, z] \mapsto [xy, xz, y^2, yz, z^2].$$

Show that $\varphi$ is a birational map onto a surface $\varphi(\mathbb{P}^2)$, and that the inverse map $\varphi(\mathbb{P}^2) \to \mathbb{P}^2$ is identified with the blowup of $\mathbb{P}^2$ at a point.