HOMEWORK #6

Due: November 3, 2021

1. (a) Show that a general hypersurface $X = \{F = 0\} \subseteq \mathbb{P}^n$, given by a degree-$d$ polynomial in $n+1$ variables, is nonsingular. (Here, as usual, "general" means this is true for all F in an open dense set of the projective space $\mathbb{P}^{N(d,n)}$ of all degree-d polynomials in $n+1$ variables.)

   (b) Show that there is an irreducible homogeneous polynomial $\Delta(d,n)$ in $\binom{n+d}{d}$ variables, unique up to nonzero scalar multiple, such a degree-d hypersurface $\{F = 0\} \subseteq \mathbb{P}^n$ is singular iff $\Delta(d,n)$ vanishes on the coefficients of F. (You may assume the base field has characteristic zero, if it helps.)

   The polynomial Δ is called the *discriminant*. (Computing it explicitly is a nontrivial problem!)

2. Let S be a surface ($\dim S = 2$) and $p \in S$ a nonsingular point. Let C and D be two curves in S passing through p, both nonsingular at p. Let $\pi\colon \widetilde{S} \to S$ be the blowup of S at p, and set $\widetilde{C} = \overline{\pi^{-1}(C \setminus p)}$ and $\widetilde{D} = \overline{\pi^{-1}(D \setminus p)}$. Let $E = \pi^{-1}(p)$. Show that $\widetilde{C} \cap E = \widetilde{D} \cap E$ if and only if C and D are *tangent* at p. (This means $T_pC = T_pD$ as subspaces of T_pS.)

3. Consider the rational map $\varphi\colon \mathbb{P}^2 \to \mathbb{P}^4$ given by
$$[x, y, z] \mapsto [xy, xz, y^2, yz, z^2].$$
   Show that φ is a birational map onto a surface $\overline{\varphi(\mathbb{P}^2)}$, and that the inverse map $\overline{\varphi(\mathbb{P}^2)} \to \mathbb{P}^2$ is identified with the blowup of $\mathbb{P}^2$ at a point.