HOMEWORK #7

Due: November 17, 2021

1. Let X be an affine algebraic set, with $\mathcal{O}(X) = A$, let $p \in X$ have maximal ideal $\mathfrak{m} \subseteq A$, and write $\mathfrak{m}_p \subseteq A_\mathfrak{m} = \mathcal{O}_{X,p}$ for its extension to the local ring. Show that there is a natural surjective homomorphism of graded k-algebras
$$\mathrm{Sym}^*(\mathfrak{m}_p/\mathfrak{m}_p^2) \twoheadrightarrow \bigoplus_{i \geq 0} \mathfrak{m}_p^i/\mathfrak{m}_p^{i+1},$$
corresponding to a closed inclusion of affine algebraic sets $C_pX \hookrightarrow T_pX$. Thus the tangent cone C_pX is intrinsic to X, and is naturally a closed subset of the Zariski tangent space.

   (Choosing an embedding $X \hookrightarrow \mathbb{A}^n$ so that $X = Z(I)$, we defined C_pX as the zeroes of the initial ideal I^{in} when $p = 0 \in \mathbb{A}^n$. We saw that $T_pX \subseteq T_p\mathbb{A}^n$ is defined by the vanishing of I^{lin}, the linear forms in I. Identify the kernel of the displayed homomorphism with $I^{\mathrm{in}}/I^{\mathrm{lin}}$.)

2. Let $p \in \mathbb{A}^2$ be a point, $X = \mathrm{Bl}_p\,\mathbb{A}^2$, and $E \subseteq X$ the exceptional divisor in the blowup. Show that $\mathrm{ord}_E(f) = \mathrm{mult}_p(f)$, for any $f \in k[x,y] \subseteq K(\mathbb{A}^2) = K(X)$. (In particular, f vanishes to order 1 on E iff p is a nonsingular point of the plane curve C_f.)

3. Consider the Stiener surface $S = \{X^2Y^2+Y^2Z^2+X^2W^2-XYZW = 0\} \subseteq \mathbb{P}^3$. Compute the singular locus of S, and determine whether S is normal.

4. Show that any algebraic curve has a plane projective model all of whose singularities have only linear branches. (See [Shafarevich, §II.5].)