1. Let $X$ be an affine algebraic set, with $\mathcal{O}(X) = A$, let $p \in X$ have maximal ideal $m \subseteq A$, and write $m_p \subseteq A_m = \mathcal{O}_{X,p}$ for its extension to the local ring. Show that there is a natural surjective homomorphism of graded $k$-algebras

$$\text{Sym}^*(m_p/m_p^2) \rightarrow \bigoplus_{i \geq 0} m_p^i/m_p^{i+1},$$

corresponding to a closed inclusion of affine algebraic sets $C_pX \hookrightarrow T_pX$. Thus the tangent cone $C_pX$ is intrinsic to $X$, and is naturally a closed subset of the Zariski tangent space.

(Choosing an embedding $X \hookrightarrow \mathbb{A}^n$ so that $X = Z(I)$, we defined $C_pX$ as the zeroes of the initial ideal $I$ in $\mathbb{A}^n$ when $p = 0 \in \mathbb{A}^n$. We saw that $T_pX \subseteq T_p\mathbb{A}^n$ is defined by the vanishing of $I$, the linear forms in $I$. Identify the kernel of the displayed homomorphism with $I/I\text{lin}$.)

2. Let $p \in \mathbb{A}^2$ be a point, $X = \text{Bl}_p \mathbb{A}^2$, and $E \subseteq X$ the exceptional divisor in the blowup. Show that $\text{ord}_E(f) = \text{mult}_p(f)$, for any $f \in k[x, y] \subseteq K(\mathbb{A}^2) = K(X)$. (In particular, $f$ vanishes to order 1 on $E$ iff $p$ is a nonsingular point of the plane curve $C_f$.)

3. Consider the Stiener surface $S = \{X^2Y^2+Y^2Z^2+X^2W^2-XYZW = 0\} \subseteq \mathbb{P}^3$. Compute the singular locus of $S$, and determine whether $S$ is normal.

4. Show that any algebraic curve has a plane projective model all of whose singularities have only linear branches. (See [Shafarevich, §II.5].)