HOMEWORK #8

Due: December 8, 2021

1. ($k = \mathbb{C}$) Let $C = C_f \subseteq \mathbb{C}^2$ be a plane curve containing $p = (0,0)$. The topology of $C \smallsetminus p$ near p is closely related to the singularity type of p in C. Specifically, the *link* of (C,p) is the intersection
$$L(p) := S^3_\epsilon \cap C \subseteq \mathbb{C}^2,$$
where S^3_ϵ is the standard sphere of radius ϵ about the origin in $\mathbb{C}^2 = \mathbb{R}^4$. For sufficiently small $\epsilon > 0$, $L(p) \subseteq S^3$ is an embedded disjoint union of S^1's (a "link"), and is independent of ϵ, up to isotopy.
   (a) If p is a nonsingular point of C, show that $L(p)$ is the "unknot", i.e., it is isotopic to the circle $a^2 + b^2 = \epsilon^2$ in $S^3_\epsilon \subseteq \mathbb{R}^4$ (with coordinates a, b, c, d).
   (b) Let $f = y^2 - x^3$, and let $C = C_f$. Check that C_f is homeomorphic to $\mathbb{C}$. Show that the link $L(p)$ is identified with the trefoil knot.
   (c) Determine $L(p)$ for $C = \{xy = 0\}$.
   (d) What is $L(p)$ if $C = \{y^2 - x^2 - x^3 = 0\}$?

2. For a divisor D on a nonsingular curve X, show that $\deg(D) = 0$ and $\ell(D) > 0 \Leftrightarrow D$ is principal.

3. For a nonsingular projective curve X, let
$$\mathrm{Cl}^0(X) = \ker(\mathrm{Cl}(X) \xrightarrow{\deg} \mathbb{Z})$$
be the degree-zero part of the divisor class group. Show that $\mathrm{Cl}^0(X)$ is trivial iff $X \cong \mathbb{P}^1$.

4. Let $C \subseteq \mathbb{P}^2$ be a cubic curve with affine equation in Weierstrass normal form $y^2 = x^3 + ax + b$. Find all points on C having order 2 in the group law.

5. (Cayley-Bacharach) If two cubic plane curves intersect in exactly 9 points, then any third cubic curve passing through 8 of these points also contains the 9th point.