HOMEWORK #2

Due: February 6, 2026

1. (Hartshorne §II.2) Exercises 3–9, 11, 13, 14, 16, 17, 18.
2. (A scheme is a functor.) Given a scheme X, define a functor
$$h_X : (\mathbf{Sch})^{\mathrm{op}} \to (\mathbf{Sets})$$
by setting $h_X(T) = \mathrm{Mor}(T, X)$, the set of morphisms from T to X. (So h_X is contravariant in T.) Verify the following:
   - The assignment $X \mapsto h_X$ is functorial: a morphism $X \to X'$ determines a natural transformation $h_X \to h_{X'}$, compatible with composition, etc. So $h_{(-)}$ is a (covariant) functor from (**Sch**) to the category of functors from (**Sch**) to (**Sets**) (whose morphisms are natural transformations).
   - There is a natural bijection $\mathrm{Mor}(X, X') \xrightarrow{\sim} \mathrm{Mor}(h_X, h_{X'})$ (functorial in both variables); that is, the functor $X \mapsto h_X$ is fully faithful.
   - Conclude that X is completely determined by the functor h_X: if $h_X \cong h_{X'}$ then $X \cong X'$.
   - Everything you checked works verbatim for (**Sch**) replaced by an arbitrary category. (This is an instance of the *Yoneda embedding.*)

   Define a functor $F_X : (\mathbf{Sch}) \to (\mathbf{Sets})$ by letting $F_X(T)$ be the set of morphisms $f : T \to X$ such that for each $t \in T$, the kernel of the homomorphism $f_t^\sharp : \mathcal{O}_{X,f(t)} \to \mathcal{O}_{T,t}$ contains the nilradical. There is an evident natural transformation $F_X \to h_X$. Show that there exists a scheme X' such that F_X is isomorphic to $h_{X'}$. (Hint: you've already done this!)

   We will see more useful applications of this principle soon.