HOMEWORK #3

Due: February 20, 2026

1. (Hartshorne §II.3) Exercises 1, 4, 5, 6, 7, 8, 11b, 12, 20.
   (Hartshorne §II.4) Exercises 1, 2, 3, 6, 9.

2. Let X be a reduced scheme of finite type over $\mathbb{C}$. By identifying the $\mathbb{C}$-points $X(\mathbb{C})$ with the closed points of X, $X(\mathbb{C})$ has an induced (subspace) topology. However, it is common to endow $X(\mathbb{C})$ with the *analytic* (or *usual*) topology, and denote it by X^{an}. Construct this space as follows.

   (a) The usual topology on $\mathbb{C}^n$ (i.e., with base of open balls $\|z-a\| < \epsilon$) is finer than the Zariski topology on $\mathbb{C}^n = \mathbb{A}^n(\mathbb{C})$.

   (b) If $X = \operatorname{Spec} A$ is affine, choose a closed embedding $X(\mathbb{C}) \hookrightarrow \mathbb{A}^n(\mathbb{C}) = \mathbb{C}^n$, and equip $X(\mathbb{C})$ with the subspace topology. Show that this topology is independent of embedding. Conclude that the topology on X^{an} is well-defined for quasi-affine X.

   (c) Suppose $f : X \to Y$ is an isomorphism of quasi-affine schemes. Show that f naturally induces a homeomorphism of spaces $X^{\mathrm{an}} \to Y^{\mathrm{an}}$.

   (d) For general X, the topology on X^{an} is defined so that for every quasi-affine open subscheme $U \subseteq X$, the subspace topology on U^{an} is the one defined above (as a subspace of some $\mathbb{C}^n$). Check that this is well-defined.

   Check that this construction is functorial: a morphism of schemes $f : X \to Y$ induces a continuous map of spaces $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$. Also check that the topology on $(X \times_{\mathbb{C}} Y)^{\mathrm{an}}$ is the product topology on $X^{\mathrm{an}} \times Y^{\mathrm{an}}$. Deduce that X^{an} is Hausdorff if X is separated. (Can you prove the converse?)

   (A similar construction also produces a sheaf of rings of holomorphic functions on X^{an}, making it a $\mathbb{C}$-analytic space.)