HOMEWORK #5

Due: March 25, 2026

1. (Projective bundles.) Recall the functor defined on schemes T over $\operatorname{Spec} A$ by
$$P^n_A(T) = \left\{\mathcal{O}_T^{\oplus n+1} \twoheadrightarrow \mathscr{L}\right\} / \cong,$$
and represented by projective space, $\mathbb{P}^n_A \cong \mathbb{P}^n \times \operatorname{Spec} A$.

   Let $\mathscr{E}$ be a locally free coherent sheaf on a scheme X, and consider the functor on X-schemes
$$P(\mathscr{E})(T) = \{f^*\mathscr{E} \twoheadrightarrow \mathscr{L}\} / \cong,$$
where $f\colon T \to X$ is the structure morphism, and isomorphisms of invertible quotients are defined as before. Show that $P(\mathscr{E})$ is represented by a proper X-scheme $\mathbb{P}(\mathscr{E}) \xrightarrow{\pi} X$, which comes equipped with a universal quotient line bundle $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)$. If $\mathscr{M}$ is a line bundle on X, show that $\mathbb{P}(\mathscr{E} \otimes \mathscr{M})$ is isomorphic to $\mathbb{P}(\mathscr{E})$, but its universal line bundle is different.

   Show that a surjection $\mathscr{E} \twoheadrightarrow \mathscr{F}$ of locally free coherent sheaves determines a closed immersion $\mathbb{P}(\mathscr{F}) \hookrightarrow \mathbb{P}(\mathscr{E})$.

   When $X = \operatorname{Spec} k$, for k an algebraically closed field, so $\mathscr{E} = V$ is a finite-dimensional vector space, how is $\mathbb{P}(V)$ related to the variety parametrizing lines in V? (The latter was identified with projective space over k last semester.)

   Extra: relax the "locally free" hypothesis to construct $\mathbb{P}(\mathscr{F})$ (with appropriate universal property) for any coherent sheaf $\mathscr{F}$.

2. (Hartshorne §II.7) 1, 2, 3, 4b, 5ade, 12.