HOMEWORK #7

Due: April 27, 2026

1. (Hartshorne §II.8) Exercises 2, 3, 5, 6.
2. (Hartshorne §III.9) Exercises 3, 7.
3. (Hartshorne §III.10) Exercise 9.
4. (Hilbert series and Hilbert polynomials.) Let $S = k[x_0, \dots, x_n]$ be equipped with the usual grading, and consider a finitely generated graded S-module M. Its *Hilbert function* is

$$d \mapsto \dim_k M_d,$$

where M_d is the dth graded component of M. The *Hilbert series* of M is the corresponding generating function:

$$H_M(t) := \sum_{d \geq 0} (\dim_k M_d)\, t^d$$

(a) Show that the Hilbert series is additive on exact sequences: if $0 \to M' \to M \to M'' \to 0$ is exact, then $H_M(t) = H_{M'}(t) + H_{M''}(t)$.

(b) Compute $H_S(t)$ explicitly, and show that $H_M(t)$ is a rational function in t for any finitely generated graded S-module.

(c) Analyzing your proof that $H_M(t)$ is rational more carefully, show that for all sufficiently large d, the Hilbert function $\dim_k M_d$ is given by a polynomial in d. (It should appear as a combination of binomial coefficients.)