59.

ON THE THEORY OF ELIMINATION.

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SUPPOSE the variables X_1, $X_2 \dots$, g in number, are connected by the h linear equations

$$\Theta_1 = \alpha_1 X_1 + \alpha_2 X_2 \dots = 0,$$
$$\Theta_2 = \beta_1 X_1 + \beta_2 X_2 \dots = 0,$$
$$\vdots$$

these equations not being all independent, but connected by the k linear equations

$$\Phi_1 = \alpha_1' \Theta_1 + \alpha_2' \Theta_2 + \dots = 0,$$
$$\Phi_2 = \beta_1' \Theta_1 + \beta_2' \Theta_2 + \dots = 0,$$
$$\vdots$$

these last equations not being independent, but connected by the l linear equations

$$\Psi_1 = \alpha_1'' \Phi_1 + \alpha_2'' \Phi_2 + \dots = 0,$$
$$\Psi_2 = \beta_1'' \Phi_1 + \beta_2'' \Phi_2 + \dots = 0,$$
$$\vdots$$

and so on for any number of systems of equations.

Suppose also that $g - h + k - l + \dots = 0$; in which case the number of quantities X_1, $X_2, \dots$ will be equal to the number of really independent equations connecting them, and we may obtain by the elimination of these quantities a result $\nabla = 0$.

To explain the formation of this final result, write

$$\nabla = \begin{array}{|c|c|l}
\hline
\begin{matrix} \alpha_1, & \beta_1 \ldots \\ \alpha_2, & \beta_2 \\ \vdots & \end{matrix} & & \\
\hline
\begin{matrix} \alpha_1', & \alpha_2' \ldots \\ \beta_1', & \beta_2' \\ \vdots & \end{matrix} & \begin{matrix} \alpha_1'', & \beta_1'' \ldots \\ \alpha_2'', & \beta_2'' \\ \vdots & \end{matrix} & \\
\hline
 & \begin{matrix} \alpha_1''', & \alpha_2''' \ldots \\ \beta_1''', & \beta_2''' \\ \vdots & \end{matrix} & \\
\hline
\end{array}$$

which for shortness may be thus represented,

$$\nabla = \begin{array}{|c|c|l}
\hline
\Omega & & \\
\hline
\Omega' & \Omega'' & \\
\hline
 & \Omega''' & \\
\hline
\end{array}$$

where Ω, Ω', Ω'', Ω''', Ω'''',... contain respectively h, h, l, l, n, n,... vertical rows, and g, k, k, m, m, p,... horizontal rows.

It is obvious, from the form in which these systems have been arranged, what is meant by speaking of a certain number of the vertical rows of Ω' and the *supplementary* vertical rows of Ω; or of a certain number of the horizontal rows of Ω'' and the *supplementary* horizontal rows of Ω', &c.

Suppose that there is only one set of equations, or $g = h$: we have here only a single system Ω, which contains h vertical and h horizontal rows, and ∇ is simply the determinant formed with the system of quantities Ω. We may write in this case $\nabla = Q$.

Suppose that there are two sets of equations, or $g = h - k$: we have here two systems Ω, Ω', of which Ω contains h vertical and $h - k$ horizontal rows, Ω' contains h vertical and k horizontal rows. From any k of the h vertical rows of Ω' form a determinant, and call this Q'; from the supplementary $h - k$ vertical rows of Ω form a determinant, and call this Q: then Q' divides Q, and we have $\nabla = Q \div Q'$.

Suppose that there are three sets of equations, or $g=h-k+l$: we have here three systems, Ω, Ω', Ω'', of which Ω contains h vertical and $h-k+l$ horizontal rows, Ω' contains h vertical and k horizontal rows, Ω'' contains l vertical and k horizontal rows. From any l of the k horizontal rows of Ω'' form a determinant, and call this Q''; from the $k-l$ supplementary horizontal rows of Ω', choosing the vertical rows at pleasure, form a determinant, and call this Q'; from the $h-k+l$ supplementary vertical rows of Ω form a determinant, and call this Q: then Q'' divides Q', this quotient divides Q, and we have $\nabla = Q \div (Q' \div Q'')$.

Suppose that there are four sets of equations, or $g=h-k+l-m$: we have here four systems, Ω, Ω', Ω'', and Ω''', of which Ω contains h vertical and $h-k+l-m$ horizontal rows, Ω' contains h vertical and k horizontal rows, Ω'' contains l vertical and k horizontal rows, and Ω''' contains l vertical and m horizontal rows. From any m of the l vertical rows of Ω''' form a determinant, and call this Q'''; from the $l-m$ supplementary vertical rows of Ω'', choosing the horizontal rows at pleasure, form a determinant, and call this Q''; from the $k-l+m$ supplementary horizontal rows of Ω', choosing the vertical rows at pleasure, form a determinant, and call this Q'; from the $h-k+l-m$ supplementary vertical rows of Ω form a determinant, and call this Q: then Q''' divides Q'', this quotient divides Q', this quotient divides Q, and $\nabla = Q \div \{Q' \div (Q'' \div Q''')\}$. The mode of proceeding is obvious.

It is clear, that if all the coefficients $\alpha, \beta, \ldots$ be considered of the order unity, ∇ is of the order $h-2k+3l-$ &c.

What has preceded constitutes the theory of elimination alluded to in my memoir "On the Theory of Involution in Geometry," *Journal*, vol. II. p. 52—61, [40]. And thus the problem of eliminating any number of variables $x, y \ldots$ from the same number of equations $U=0$, $V=0, \ldots$ (where $U, V, \ldots$ are homogeneous functions of any orders whatever) is completely solved; though, as before remarked, I am not in possession of any method of arriving *at once* at the final result in its most simplified form; my process, on the contrary, leads me to a result encumbered by an extraneous factor, which is only got rid of by a number of successive divisions less by two than the number of variables to be eliminated.

To illustrate the preceding method, consider the three equations of the second order,

$$U = a\;x^2 + b\,y^2 + c\;z^2 + l\;yz + m\;zx + n\;xy = 0,$$
$$V = a'\,x^2 + b'y^2 + c'\,z^2 + l'\,yz + m'\,zx + n'\,xy = 0,$$
$$W = a''x^2 + b''y^2 + c''z^2 + l''yz + m''zx + n''xy = 0.$$

Here, to eliminate the fifteen quantities x^4, y^4, z^4, y^3z, z^3x, x^3y, yz^3, zx^3, xy^3, y^2z^2, z^2x^2, x^2y^2, x^2yz, y^2zx, z^2xy, we have the eighteen equations

$$x^2U = 0,\quad y^2U = 0,\quad z^2U = 0,\quad yzU = 0,\quad zxU = 0,\quad xyU = 0,$$
$$x^2V = 0,\quad y^2V = 0,\quad z^2V = 0,\quad yzV = 0,\quad zxV = 0,\quad xyV = 0,$$
$$x^2W = 0,\quad y^2W = 0,\quad z^2W = 0,\quad yzW = 0,\quad zxW = 0,\quad xyW = 0,$$

equations, however, which are not independent, but are connected by

$$a''x^2V + b''y^2V + c''z^2W + l''yzV + m''zxV + n''xyV$$
$$- (a'x^2W + b'y^2W + c'z^2W + l'yzW + m'zxW + n'xyW) = 0,$$

$$ax^2W + by^2W + cz^2W + lyzW + mzxW + nxyW$$
$$- (a''x^2U + b''y^2U + c''z^2U + l''yzU + m''zxU + n''xyU) = 0,$$

$$a'x^2U + b'y^2U + c'z^2U + l'yzU + m'zxU + n'xyU$$
$$- (ax^2V + by^2V + cz^2V + lyzV + mzxV + nxyV) = 0.$$

Arranging these coefficients in the required form, we have the following value of ∇.

a						a'						a''					
	b						b'						b''				
		c						c'						c''			
	l		b				l'		b'				l''		b''		
		m		c				m'		c'				m''		c''	
n					a	n'					a'	n''					a''
		l	c					l'	c'					l''	c''		
m				a		m'				a'		m''				a''	
	n				b		n'				b'		n''				b''
	c	b	l				c'	b'	l'				c''	b''	l''		
c		a		m		c'		a'		m'		c''		a''		m''	
b	a				n	b'	a'				n'	b''	a''				n''
l			a	n	m	l'			a'	n'	m'	l''			a''	n''	m''
	m		n	b	l		m'		n'	b'	l'		m''		n''	b''	l''
		n	m	l	c			n'	m'	l'	c'			n''	m''	l''	c''
						a''	b''	c''	l''	m''	n''	$-a'$	$-b'$	$-c'$	$-l'$	$-m'$	$-n'$
$-a''$	$-b''$	$-c''$	$-l''$	$-m''$	$-n''$							a	b	c	l	m	n
a'	b'	c'	l'	m'	n'	$-a$	$-b$	$-c$	$-l$	$-m$	$-n$						

which may be represented as before by

$$\nabla = \left|\begin{array}{c} \Omega \\ \hline \Omega' \end{array}\right| .$$

Thus, for instance, selecting the first, second, and sixth lines of Ω' to form the determinant Q', we have $Q' = a''(a'b'' - a''b')$; and then Q must be formed from the third, fourth, fifth, seventh, &c.... eighteenth lines of Ω. (It is obvious that if Q' had been formed from the first, second, and third lines of Ω', we should have had $Q' = 0$; the corresponding value of Q would also have vanished, and an illusory result be obtained; and similarly for several other combinations of lines.)