ANSWERS TO PROBLEM 9 OF SECTION 5

Before we answer the question, let us look at a preliminary result.

Suppose that a single trial of an experiment results in either success with probability \( p \) or failure with probability \( q \). We must have \( q = 1 - p \) (why?). The experiment is performed with successive independent trials until the first success occurs. An outcome of such a sequence of trials can be expressed by a sequence of \( f \)'s followed by an \( s \). For example \( ffffs \) represents after 5 failures we get a success. Let \( X \) denote the number of failures before the first success. So \( X(ffffs) = 5 \). \( X \) is a random variable. We have

\[
\{ X = 0 \} = \{ s \}, \quad \{ X = 1 \} = \{ fs \}, \quad \{ X = 2 \} = \{ ffs \}, \quad \{ X = 3 \} = \{ fffs \}, \quad \{ X = 4 \} = \{ ffffs \}, \quad \ldots
\]

The probabilities are

\[
P(X = 0) = P(\{ s \}) = p, \quad P(X = 1) = P(\{ fs \}) = P(\{ f \})P(\{ s \}) = qp, \quad P(X = 2) = P(\{ ffs \}) = qqp = q^2p,
\]

\[
P(X = 3) = P(\{ fffs \}) = qqqp = q^3p, \quad P(X = 4) = P(\{ ffffs \}) = qqqqp = q^4p, \quad \ldots ,
\]

\[
P(X = k) = P(\{ ff...fs \}) = qqq...qp = q^k p \text{ (} k \text{ } f\text{'s and } k \text{ } q\text{'s), } \ldots
\]

(This is the so called geometric distribution.)

**Solution.** Since the one who starts the game (i.e. who throw the dice first) has advantage to win, he should put a little more money in the pot in order to have a fair game. (The game is fair if the expected returns for both players are equal.) Suppose he should put an extra \( \$c \) in the pot. We are going to compute the expected returns for each player.

Consider the single trial of throwing a pair of dice. It is a success (i.e. \( \{ s \} \)) if the sum is 5, and it is a failure (i.e. \( \{ f \} \)) if the sum is not 5. By a simple computation we have

\[
p = P(\{ s \}) = \frac{4}{6 \cdot 6} = \frac{1}{9}, \quad q = P(\{ f \}) = 1 - p = \frac{8}{9}.
\]

Since “one wins” and “the other loses” are the same meaning, we have the following

\[
\{ \text{the first player wins} \} = \{ \text{the second player loses} \} = \{ s, ffs, ffffs, ffffffs, } \ldots
\]

\[
\{ \text{the first player loses} \} = \{ \text{the second player wins} \} = \{ fs, ffs, ffffs, ffffffs, } \ldots
\]

and so

\[
P(\{ \text{the first player wins} \}) = P(\{ s \}) + P(\{ ffs \}) + P(\{ ffffs \}) + \cdots
\]

\[
= p + q^2p + q^3p + q^4p + \cdots
\]

\[
= p(1 + q^2 + q^4 + q^6 + \cdots)
\]

\[
= \frac{p}{1 - q^2} = \frac{1/9}{1 - (8/9)^2} = 9/17.
\]

Using \( P(A') = 1 - P(A) \), we have \( P(\{ \text{the first player loses} \}) = 1 - 9/17 = 8/17 = P(\{ \text{the second player wins} \}) \), and \( P(\{ \text{the second player loses} \}) = P(\{ \text{the first player wins} \}) = 9/17 \).

Let \( R_1 \) be the returns player 1, and \( R_2 \) the returns player 2. Then \( R_1(\omega) = 1 \) or \( R_1(\omega) = -(1 + c) \) if \( \omega \) means player 1 wins or loses respectively, and \( R_2(\omega) = 1 + c \) or \( -1 \) if \( \omega \) means player 2 wins or loses respectively. Then \(^1\)

\[
E(R_1) = 1 \cdot P(\text{player 1 wins}) - (1 + c) \cdot P(\text{player 1 loses}) = 1 \cdot \frac{9}{17} - (1 + c) \cdot \frac{8}{17}
\]

\[
E(R_2) = (1 + c) \cdot P(\text{player 2 wins}) - 1 \cdot P(\text{player 2 loses}) = (1 + c) \cdot \frac{8}{17} - 1 \cdot \frac{9}{17}.
\]

Then “game is fair” \( \Rightarrow E(R_1) = E(R_2) \Rightarrow \frac{9}{17} - (1 + c) \frac{8}{17} = (1 + c) \frac{8}{17} - \frac{9}{17} \Rightarrow c = \frac{1}{8} \).

\(^1\)Note that \( E(R) = \sum_r r P(R = r) \), the sum of “value of \( R \) x probability of \( R \) equal to that value”.