**CONDITIONAL PROBABILITIES**

\[ P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(A \cap B) = P(A|B) \cdot P(B), \quad P(A'|B) = 1 - P(A|B). \]

Let \( B_1, B_2, \ldots, B_n \) be a partition of \( S \). Then

\[ P(A) = P(A \cap B_1) + \cdots + P(A \cap B_n) = P(A|B_1) \cdot P(B_1) + \cdots + P(A|B_n) \cdot P(B_n), \]

\[ P(B_i|A) = \frac{P(A|B_i) \cdot P(B_i)}{P(A)} = \frac{P(A|B_i) \cdot P(B_i)}{P(A|B_1) \cdot P(B_1) + \cdots + P(A|B_n) \cdot P(B_n)} \]

Especially, (since \( B \) and \( B' \) form a partition of \( S \))

\[ P(A) = P(A \cap B) + P(A \cap B'), \quad P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)} = \frac{P(A|B) \cdot P(B)}{P(A|B) \cdot P(B) + P(A|B') \cdot P(B')} \]

**Examples**

1. Three marksmen hit the target with probabilities 1/2, 2/3, 3/4, respectively. They shoot simultaneously and there are two hits. Who missed? Find the probabilities.

   **Solution.** Let \( A, B, \) and \( C \) be the events that the first, the second and the third one hits respectively, and let \( T \) be the event that there are two hits. \( A, B, \) and \( C \) are independent, and \( T = (A' \cap B \cap C) \cup (A \cap B' \cap C) \cup (A \cap B \cap C') \), a disjoint union. We have

   \[ P(T) = P(A' \cap B \cap C) + P(A \cap B' \cap C) + P(A \cap B \cap C') = P(A')P(B)P(C) + P(A)P(B')P(C) + P(A)P(B)P(C'). \]

   \[ = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} = \frac{11}{24}. \]

   We want to find \( P(A'|T), P(B'|T), \) and \( P(C'|T) \).

   **\( P(A'|T) \):** Note that \( A' \cap T = A' \cap B \cap C \) (why?). We have

   \[ P(A'|T) = \frac{P(A' \cap T)}{P(T)} = \frac{P(A')P(B)P(C)}{P(T)} = \frac{\frac{3}{2} \cdot \frac{2}{3} \cdot \frac{3}{4}}{\frac{11}{24}} = \frac{6}{11}. \]

   Similarly, we have

   \[ P(B'|T) = \frac{3}{11}, \quad P(C'|T) = \frac{2}{11}. \]

2. Telegraphic signals “dot” and “dash” are sent in the proportion 3:4. Due to conditions causing very erratic transmission, a dot becomes a dash with probability 1/4, whereas a dash becomes a dot with probability 1/3. If a dot is received, what is the probability that it was sent as a dot?

   **Solution.** Let \( A \) be the event that a dot is received, and \( B \) a dot is sent. We want to find \( P(B|A) \). Note that \( P(B) = 3/7, P(B') = 4/7, P(A|B) = 3/4, \) and \( P(A|B') = 1/3 \), we have

   \[ P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A|B) \cdot P(B) + P(A|B') \cdot P(B')} = \frac{\frac{3}{4} \cdot \frac{3}{7} + \frac{1}{3} \cdot \frac{3}{2}}{\frac{3}{4} \cdot \frac{3}{7} + \frac{1}{3} \cdot \frac{3}{2}} = \frac{189}{301} = \frac{27}{43}. \]
3. First throw a fair die, then throw as many fair coins as the point shown on the die.

(a) Find the probability of getting \(k\) heads.

(b) If 3 heads are obtained, what is the probability that the die showed \(m\)?

**Solution.** Let \(D_n\) be the event that the die showed \(n\) and \(H_k\) be the event that \(k\) heads were obtained.

(a) Since \(D_1, ..., D_6\) form a partition,

\[
P(H_k) = P(H_k \cap D_1) + \cdots + P(H_k \cap D_6) = \sum_{n=1}^{6} P(H_k \cap D_n) = \sum_{n=1}^{6} P(H_k | D_n) \cdot P(D_n).
\]

First, note that \(P(D_n) = 1/6\) for any \(1 \leq n \leq 6\). Now, if \(n < k\), less than \(k\) coins cannot show \(k\) heads, so \(P(H_k | D_n) = 0\). If \(n \geq k\), then \(P(H_k | D_n) = \binom{n}{k} \frac{1}{2^n}\). So

\[
P(H_k | D_n) \cdot P(D_n) = \begin{cases} 0 & n < k \\ \frac{1}{6} \binom{n}{k} \frac{1}{2^n} & n \geq k \end{cases}
\]

where \(\binom{n}{k} = \frac{n!}{k!(n-k)!}\). Substituting (2) into (1), we get

\[
P(H_k) = \sum_{n=k}^{6} \frac{1}{6} \binom{n}{k} \frac{1}{2^n} = \frac{1}{6} \sum_{n=k}^{6} \binom{n}{k} \frac{1}{2^n}.
\]

(b) We need \(P(D_m | H_3)\). If \(m < 3\), less than 3 coins could not give 3 heads, so \(P(D_m | H_3) = 0\). If \(m \geq 3\), turning around and by (2) we have

\[
P(D_m | H_3) = \frac{P(H_3 | D_m) \cdot P(D_m)}{P(H_3)} = \frac{\frac{1}{6} \binom{3}{m} \frac{1}{2^m}}{\frac{1}{6} \sum_{n=3}^{6} \binom{n}{3} \frac{1}{2^n}} = \frac{\binom{3}{m} \frac{1}{2^m}}{\sum_{n=3}^{6} \binom{n}{3} \frac{1}{2^n}}
\]

So,

\[
P(D_m | H_3) = \begin{cases} 0 & m < 3 \\ \frac{\binom{3}{m} \frac{1}{2^m}}{\sum_{n=3}^{6} \binom{n}{3} \frac{1}{2^n}} & 3 \leq m \leq 6 \end{cases}
\]

4. Given that a throw of three fair dice shows different faces, what is the probability that (a) at least one is six; (b) the total is eight?

**Solution.** We use \((l, m, n)\) to represent the outcomes of throwing three dice where \(l\) is the number shown on the first die, \(m\) is the number shown on the second die, ... . Then \(S\) has \(6^3\) different outcomes. Let \(B\) be the event of three different faces. Then \(B = \{(l, m, n) \mid l \neq m, m \neq n, n \neq l\}\). Note that \(B\) contains \(6 \cdot 5 \cdot 4 = 120\) elements.

(a) Let \(A\) be the event that at least one is six. We need to find \(P(A | B)\). Now \(A'\) is the event that none is six. So, \(A' = \{(l, m, n) \mid l < 6, m < 6, n < 6\}\). \(A' \cap B\) contains \(5 \cdot 4 \cdot 3 = 60\) elements, and so

\[
P(A | B) = 1 - P(A' | B) = 1 - \frac{P(A' \cap B)}{P(B)} = 1 - \frac{60/6^3}{120/6^3} = 1 - 1/2 = 1/2.
\]

(b) Let \(C\) be the event that the total is eight. So \(C = \{(l, m, n) \mid l + m + n = 8\}\). We need \(P(C | B)\).

\[
P(C | B) = \frac{P(C \cap B)}{P(B)} = \text{... (counting)} = \frac{12}{120} = \frac{1}{10}.
\]