Chapter 10
Binomial Option Pricing: I

Question 10.1.

Using the formulas given in the main text, we calculate the following values:

a) for the European call option:

$$\Delta = 0.5$$
$$B = -38.4316$$
$$price = 11.5684$$

b) for the European put option:

$$\Delta = -0.5$$
$$B = 62.4513$$
$$price = 12.4513$$

Question 10.2.

a) Using the formulas of the textbook, we obtain the following results:

$$\Delta = 0.7$$
$$B = -53.8042$$
$$price = 16.1958$$

b) If we observe a price of \$17, then the option price is too high relative to its theoretical value. We sell the option and synthetically create a call option for \$19.196. In order to do so, we buy 0.7 units of the share and borrow \$53.804. These transactions yield no risk and a profit of \$0.804.

c) If we observe a price of \$15.50, then the option price is too low relative to its theoretical value. We buy the option and synthetically create a short position in an option. In order to do so, we sell 0.7 units of the share and lend \$53.8042. These transactions yield no risk and a profit of \$0.696.

Question 10.3.

a) Using the formulas of the textbook, we obtain the following results:

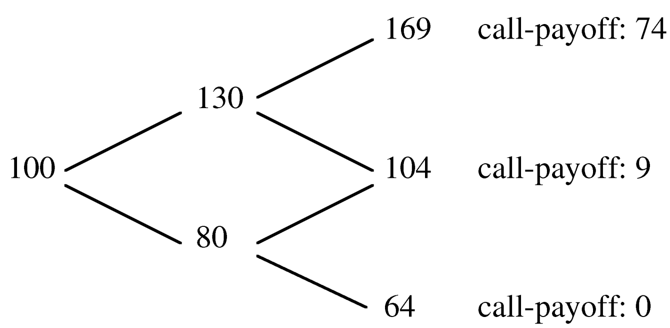
$$\Delta = -0.3$$
$$B = 37.470788$$
$$price = 7.4707$$

b) If we observe a price of \$8, then the option price is too high relative to its theoretical value. We sell the option and synthetically create a long put option for \$7.471. In order to do so, we sell 0.3 units of the share and lend \$37.471. These transactions yield no risk and a profit of \$0.529.

c) If we observe a price of \$6, then the put option price is too low relative to its theoretical value. We buy the option and synthetically create a short position in the option. In order to do so, we buy 0.3 units of the share and borrow \$37.471. These transactions yield no risk and a profit of \$1.471.

Question 10.4.

The stock prices evolve according to the following picture:



Since we have two binomial steps, and a time to expiration of one year, h is equal to 0.5. Therefore, we can calculate with the usual formulas for the respective nodes:

$t = 0, S = 100$	$t = 1, S = 80$	$t = 1, S = 130$
$\Delta = 0.691$	$\Delta = 0.225$	$\Delta = 1$
$B = -49.127$	$B = -13.835$	$B = -91.275$
$price = 19.994$	$price = 4.165$	$price = 38.725$

Question 10.5.

$S(0) = 80$:

	$t = 0, S = 80$	$t = 1, S = 64$	$t = 1, S = 104$
delta	0.4651	0	0.7731
B	−28.5962	0	−61.7980
premium	8.6078	0	18.6020

$S(0) = 90$:

	$t = 0, S = 90$	$t = 1, S = 72$	$t = 1, S = 117$
delta	0.5872	0	0.9761
B	−40.6180	0	−87.7777
premium	12.2266	0	26.4223

$S(0) = 110$:

	$t = 0, S = 110$	$t = 1, S = 88$	$t = 1, S = 143$
delta	0.7772	0.4409	1
B	−57.0897	−29.8229	−91.2750
premium	28.4060	8.9771	51.7250

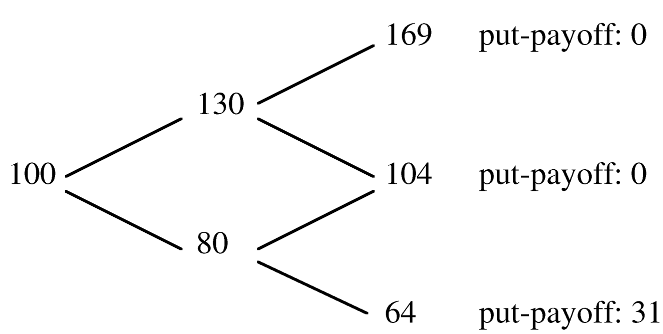
$S(0) = 120$:

	$t = 0, S = 120$	$t = 1, S = 96$	$t = 1, S = 156$
delta	0.8489	0.6208	1
B	−65.0523	−45.8104	−91.2750
premium	36.8186	13.7896	64.7250

As the initial stock price increases, the 95-strike call option is increasingly in the money. With everything else equal, it is more likely that the option finishes in the money. A hedger, e.g., a market maker, must therefore buy more and more shares initially to be able to cover the obligation she will have to meet at expiration. This number of shares in the replicating portfolio is measured by delta. The initial call delta thus increases when the initial stock price increases.

Question 10.6.

The stock prices evolve according to the following picture:



Since we have two binomial steps, and a time to expiration of one year, h is equal to 0.5. Therefore, we can calculate with the usual formulas for the respective nodes:

$t = 0, S = 100$	$t = 1, S = 80$	$t = 1, S = 130$
$\Delta = -0.3088$	$\Delta = -0.775$	$\Delta = 0$
$B = 38.569$	$B = 77.4396$	$B = 0$
$price = 7.6897$	$price = 15.4396$	$price = 0$

Question 10.7.

$S(0) = 80$:

	$t = 0, S = 80$	$t = 1, S = 64$	$t = 1, S = 104$
delta	−0.5350	−1	−0.2269
B	59.0998	91.275	29.4770
premium	16.3039	27.275	5.8770

$S(0) = 90$:

	$t = 0, S = 90$	$t = 1, S = 72$	$t = 1, S = 117$
delta	−0.4128	−1	−0.0239
B	47.0781	91.275	3.4973
premium	9.9226	19.275	0.6973

$S(0) = 110$:

	$t = 0, S = 110$	$t = 1, S = 88$	$t = 1, S = 143$
delta	−0.2228	−0.5591	0
B	30.6064	61.4521	0
premium	6.1022	12.2521	0

$S(0) = 120$:

	$t = 0, S = 120$	$t = 1, S = 96$	$t = 1, S = 156$
delta	−0.1511	−0.3792	0
B	22.6437	45.4646	0
premium	4.5146	9.0646	0

$S(0) = 130$:

	$t = 0, S = 130$	$t = 1, S = 104$	$t = 1, S = 169$
delta	−0.0904	−0.2269	0
B	14.6811	29.4770	0
premium	2.9271	5.8770	0

As the initial stock price increases, the 95-strike put option is increasingly out of the money. With everything else equal, it is more likely that the option finishes out of the money. A hedger, e.g., a market maker, must therefore sell fewer shares initially to be able to cover the obligation she will have to meet at expiration. This number of shares in the replicating portfolio is measured by delta. The initial put delta thus tends towards zero when the initial stock price increases.

Question 10.8.

We must compare the results of the equivalent European put that we calculated in exercise 10.6. with the value of immediate exercise. In 10.6., we calculated:

$t = 1, S = 80$	$t = 1, S = 130$
$\Delta = -0.775$	$\Delta = 0$
$B = 77.4396$	$B = 0$
$price = 15.4396$	$price = 0$
immediate exercise =	immediate exercise
$\max(95 - 80, 0) = 15$	$= \max(95 - 130, 0) = 0$

Since the value of immediate exercise is smaller than or equal to the continuation value (of the European options) at both nodes of the tree, there is no benefit to exercising the options before expiration. Therefore, we use the European option values when calculating the $t = 0$ option price:

$$t = 0, S = 100$$
$$\Delta = -0.3088$$
$$B = 38.569$$
$$price = 7.6897$$
$$\text{immediate exercise} = \max(95 - 100, 0) = 0$$

Since the option price is again higher than the value of immediate exercise (which is zero), there is no benefit to exercising the option at $t = 0$. Since it is never optimal to exercise earlier, the early exercise option has no value. The value of the American put option is identical to the value of the European put option.

Question 10.9.

a) We can calculate the option delta, B and the premium with our standard binomial pricing formulas:

$$\Delta = 1$$
$$B = -46.296$$
$$price = 53.704$$

It is no problem to have a d that is larger than one. The only restriction that we have imposed is that $d < e^{(r-\delta)h} = e^{(0.07696)1} = 1.08$, which is respected.

b) We may expect the option premium to go down drastically, because with a d equal to 0.6, the option is only slightly in the money in the down state at $t = 1$. However, the potential in the up state is even higher, and it is difficult to see what effect the change in u and d has on the risk-neutral probability. Let's have a look at put-call-parity. The key is the put option. A put option with a strike of 50 never pays off, neither in a) nor in b), because in a), the lowest possible stock price is 105, and in b), it is 50. Therefore, the put option has a value of zero. But then, the put-call-parity reduces to:

$$C = S - Ke^{-0.07696} = 100 - 50 \times 0.926 = 53.704.$$

Clearly, as long as the strike price is inferior to the lowest value the stock price can attain at expiration, the value of the call option is independent of u and d. Indeed, we can calculate:

$$\Delta = 1$$
$$B = -46.296$$
$$price = 53.704$$

c) Again, we are tempted to think in the wrong direction. You may think that, since the call option can now expire worthless in one state of the world, it must be worth less than in part b). This is not correct. Let us use put-call-parity to see why.
Now, with $d = 0.4$, a stock price of 40 at $t = 1$ is admissible, and the corresponding put option has a positive value, because it will pay off in one state of the world. We can use put-call-parity to see that:

$$C = S - Ke^{-0.07696} + P = 100 - 50 \times 0.926 + P = 53.704 + P > 53.704.$$

Indeed, we can calculate:

$$\Delta = 0.9$$
$$B = -33.333$$
$$price = 56.6666$$

Question 10.10.

a) We can calculate for the different nodes of the tree:

	node uu	node ud = du	node dd
delta	1	0.8966	0
B	−92.5001	−79.532	0
call premium	56.6441	15.0403	0
value of early exercise	54.1442	10.478	0

Using these values at the previous node and at the initial node yields:

	$t = 0$	node d	node u
delta	0.7400	0.4870	0.9528
B	−55.7190	−35.3748	−83.2073
call premium	18.2826	6.6897	33.1493
value of early exercise	5	0	27.1250

Please note that in all instances the value of immediate exercise is smaller than the continuation value, the (European) call premium. Therefore, the value of the European call and the American call are identical.

b) We can calculate similarly the binomial prices at each node in the tree. We can calculate for the different nodes of the tree:

	node uu	node $ud = du$	node dd
delta	0	−0.1034	−1
B	0	12.968	92.5001
put premium	0	2.0624	17.904
value of early exercise	0	0	20.404

Using these values at the previous node and at the initial node yields:

	$t = 0$	node d	node u
delta	−0.26	−0.513	−0.047
B	31.977	54.691	6.859
put premium	5.979	10.387	1.091
value of early exercise	0	8.6307	0

c) From the previous tables, we can see that at the node dd, it is optimal to early exercise the American put option, because the value of early exercise exceeds the continuation value. Therefore, we must use the value of 20.404 in all relevant previous nodes when we back out the prices of the American put option. We have for nodes d and 0 (the other nodes remain unchanged):

	$t = 0$	node d
delta	−0.297	−0.594
B	36.374	63.005
put premium	6.678	11.709
value of early exercise	0	8.6307

The price of the American put option is indeed 6.678.

Question 10.11.

a) We can calculate for the different nodes of the tree, taking into account the dividend yield:

	node uu	node $ud = du$	node dd
delta	0.974	0.6687	0
B	−92.5001	−56.239	0
call premium	45.1773	10.635	0
value of early exercise	46.398	5	0

We can see that for the node uu, the value of early exercise exceeds the continuation value. In this case, we exercise the American option early if we are at the node uu, and the value of the American call and the European call option will differ.
We have for the European call option:

	$t = 0$	node d	node u
delta	0.587	0.354	0.8124
B	−44.760	−25.014	−70.887
call premium	13.941	4.7304	25.719
value of early exercise	5	0	23.911

and for the American call option:

	$t = 0$	node d	node u
delta	0.602	0.354	0.841
B	−46.037	−25.014	−73.759
call premium	14.183	4.7304	26.262
value of early exercise	5	0	23.911

b) We can calculate similarly the binomial prices at each node in the tree for the put option:

	node uu	node $ud = du$	node dd
delta	0	−0.3049	−0.9737
B	0	36.262	92.500
put premium	0	5.767	23.639
value of early exercise	0	0	24.278

Using those put premium values at the previous nodes and at the initial node yields:

	$t = 0$	node d	node u
delta	−0.336	−0.594	−0.136
B	42.936	65.052	19.179
put premium	9.326	15.068	3.05
value of early exercise	0	10.9035	0

The price of the European put option is: 9.326.

c) From the previous tables, we can see that at the node dd, it is optimal to early exercise the American put option, because the value of early exercise exceeds the continuation value. Therefore, we must use the value of 24.278 in all relevant previous nodes when we back out the prices of the American put option. We have for nodes d and 0 (the other nodes remain unchanged):

	$t = 0$	node d
delta	−0.346	−0.616
B	44.06	67.177
put premium	9.5046	15.406
value of early exercise	0	10.903

The price of the American put option is 9.5046.

Question 10.12.

a) We can calculate u and d as follows:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.08)\times 0.25+0.3\times\sqrt{0.25}} = 1.1853$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.08)\times 0.25-0.3\times\sqrt{0.25}} = 0.8781$$

b) We need to calculate the values at the relevant nodes in order to price the European call option:

	$t = 0$	node d	node u
delta	0.6074	0.1513	1
B	−20.187	−4.5736	−39.208
call premium	4.110	0.7402	8.204

c) We can calculate at the relevant nodes (or, equivalently, you can use put-call-parity for the European put option):

European put	$t = 0$	node d	node u
delta	−0.3926	−0.8487	0
B	18.245	34.634	0
put premium	2.5414	4.8243	0

For the American put option, we have to compare the premia at each node with the value of early exercise. We see from the following table that at the node d, it is advantageous to exercise the option early; consequently, we use the value of early exercise when we calculate the value of the put option.

American put	$t = 0$	node d	node u
delta	−0.3968	−0.8487	0
B	18.441	34.634	0
put premium	2.5687	4.8243	0
value of early exercise	0	4.8762	0

Question 10.13.

a) This question deals with the important issue of rebalancing a replicating portfolio. From the previous exercise, part a), we calculate delta and B of the call option. We obtained:

	$t = 0$	node d	node u
delta	0.6074	0.1513	1
B	−20.187	−4.5736	−39.208
call premium	4.110	0.7402	8.204

Therefore, at time $t = 0$, we will buy 0.6074 shares of the stock and borrow \$20.187 from the bank. This will cost us \$4.110, and our proceeds from the sold option are \$5, which yields a total profit of \$0.89.

b) Suppose that in the next period, we are in state u (without loss of generality). At that point, the stock price is $u \times S_0 = 1.1853 \times 40 = 47.412$. Since we assume that the call is fairly priced, we can buy a call to offset our written call for \$8.204. We sell our 0.6074 shares for \$28.798, and we pay back the money we borrowed, plus accrued interest: $20.187 \times e^{0.08 \times 0.25} = 20.5948$. Thus, our total cash flow is \$28.798 − \$8.204 − \$20.5948 ≈ 0 (small differences due to rounding). We see that we were perfectly hedged, and have no cash outflow in period 1.

If the option continued to be overpriced, we would have to change the replicating portfolio according to the above table (i.e., in state u, we would buy an additional $1 - 0.6074$ shares and take on an additional loan of $39.208 - 20.187$ to finance it) and stick with our option until the final period. In the final period, the payoff from the option exactly offsets our obligation from the hedging position, and again, there would be no cash outflow.

c) If the option were underpriced, we would liquidate our position as in part b), but could make an additional profit, because we could buy the offsetting option for less than it is worth. Even better, if we could buy more than one option at the advantageous price, we could build up another arbitrage position, entering into a position where we buy the cheap option and replicate the short position synthetically.

Question 10.14.

a) We can calculate the price of the call currency option in a very similar way to our previous calculations. We simply replace the dividend yield with the foreign interest rate in our formulas. Thus, we have:

	node uu	node $ud = du$	node dd
delta	0.9925	0.9925	0.1964
B	−0.8415	−0.8415	−0.1314
call premium	0.4734	0.1446	0.0150

Using these call premia at all previous nodes yields:

	$t = 0$	node d	node u
delta	0.7038	0.5181	0.9851
B	−0.5232	−0.3703	−0.8332
call premium	0.1243	0.0587	0.2544

The price of the European call option is $0.1243.

b) For the American call option, the binomial approach yields:

	node uu	node $ud = du$	node dd
delta	0.9925	0.9925	0.1964
B	−0.8415	−0.8415	−0.1314
call premium	0.4734	0.1446	0.0150
value of early exercise	0.4748	0.1436	0

Using the maximum of the call premium and the value of early exercise at the previous nodes and at the initial node yields:

	$t = 0$	node d	node u
delta	0.7056	0.5181	0.9894
B	−0.5247	−0.3703	−0.8374
call premium	0.1245	0.0587	0.2549
value of early exercise	0.07	0	0.2540

The price of the American call option is: $0.1245.

Question 10.15.

a) We can calculate the price of the currency option in a very similar way to our previous calculations. We simply replace the dividend yield with the foreign interest rate in our formulas. Thus, we have:

	node uu	node $ud = du$	node dd
delta	0	−0.352	−0.993
B	0	0.419	0.99
put premium	0	0.069	0.2504

Using these put premia at all previous nodes yields:

	$t = 0$	node d	node u
delta	−0.509	−0.725	−0.207
B	0.605	0.7870	0.273
put premium	0.13678	0.1865	0.0449
value of early exercise	0	0.172	0

b) For the American put option, this yields:

	node uu	node $ud = du$	node dd
delta	0	−0.352	−0.993
B	0	0.419	0.99
put premium	0	0.069	0.2504
value of early exercise	0	0.0064	0.2548

Using the maximum of the put premium and the value of early exercise at the previous nodes and at the initial node yields:

	$t = 0$	node d	node u
delta	−0.455	−0.651	−0.181
B	0.617	0.8043	0.273
put premium	0.1386	0.1894	0.045
value of early exercise	0.08	0.172	0

The price of the American put option is: $0.1386.

Question 10.16.

aa) We now have to inverse the interest rates: We have a Yen-denominated option, therefore, the dollar interest rate becomes the foreign interest rate. With these changes, and equipped with an exchange rate of Y120/$ and a strike of Y120, we can proceed with our standard binomial procedure.

	node uu	node $ud = du$	node dd
delta	0.9835	0.1585	0
B	−119.6007	−17.4839	0
call premium	9.3756	1.0391	0

Using these call premia at all previous nodes yields:

	$t = 0$	node d	node u
delta	0.3283	0.0802	0.5733
B	−36.6885	−8.4614	−66.8456
call premium	2.7116	0.5029	5.0702

The price of the European Yen-denominated call option is $2.7116.

ab) For the American call option, the binomial approach yields:

	node uu	node $ud = du$	node dd
delta	0.9835	0.1585	0
B	−119.6007	−17.4839	0
call premium	9.3756	1.0391	0
value of early exercise	11.1439	0	0

Using the maximum of the call premium and the value of early exercise at the previous nodes and at the initial node yields:

	$t = 0$	node d	node u
delta	0.3899	0.0802	0.6949
B	−43.6568	−8.4614	−81.2441
call premium	3.1257	0.5029	5.9259
value of early exercise	0	0	5.4483

b) For the Yen-denominated put option, we have:

	node uu	node $ud = du$	node dd
delta	0	−0.8249	−0.9835
B	0	102.1168	119.6007
put premium	0	5.7287	17.2210
value of early exercise	0	3.1577	15.8997

We can clearly see that early exercise is never optimal at those stages. We can therefore calculate at the previous nodes:

	$t = 0$	node d	node u
delta	−0.6229	−0.8870	−0.3939
B	82.1175	110.7413	52.3571
put premium	7.37	11.602	2.9372
value of early exercise	0	8.2322	0

We can see that the American and the European put option must have the same price, since it is never optimal to exercise the American put option early. The price of the put option is 7.37.

c) The benefit of early exercise for a put option is to receive the strike price earlier on and start earning interest on it. The cost associated with early exercising a put is to stop earning income on the asset we give up. In this case, the strike is 120 Yen, and the Yen interest rate is not very favorable compared to the dollar interest rate. We would give up a high yield instrument and receive a low yield instrument when we early exercise the put option. This is not beneficial, and it is reflected by the non-optimality of early exercise of the put option.

For the call option, the opposite is true: When exercising the call option, we receive a dollar and give up 120 Yen. Therefore, we receive the high-yield instrument, and if the exchange rate moves in our favor, we want to exercise the option before expiration.

Question 10.17.

We have to pay attention when we calculate u and d. We must use the formulas given in the section options on futures contracts of the main text. In particular, we must remember that, while it is possible to calculate a delta, the option price is just the value of B, because it does not cost anything to enter into a futures contract.

We calculate:

$$u = e^{\sigma\sqrt{h}} = e^{0.1\sqrt{1}} = 1.1052$$
$$d = e^{-\sigma\sqrt{h}} = e^{-0.1\sqrt{1}} = 0.9048$$

Now, we are in a position to calculate the option's delta and B, and thus the option price. We have:

delta	0.6914
B	18.5883
premium	18.5883

This example clearly shows that the given argument is not correct. As it costs nothing to enter into the futures contract, we would not have to borrow anything if the statement was correct. We do not borrow to buy the underlying asset.

Rather, we borrow exactly the right amount so that we can, together with the position in the underlying asset, replicate the payoff structure of the call option in the future (remember that we initially solved the system of two equations).

Question 10.18.

a) We have to use the formulas of the textbook to calculate the stock tree and the prices of the options. Remember that while it is possible to calculate a delta, the option price is just the value of B, because it does not cost anything to enter into a futures contract. In particular, this yields the following prices: For the European call and put, we have: $premium = 122.9537$. The prices must be equal due to put-call-parity.

b) We can calculate for the American call option: $premium = 124.3347$ and for the American put option: $premium = 124.3347$.

c) We have the following time 0 replicating portfolios:

For the European call option:

Buy 0.5371 futures contracts.
Borrow 122.9537

For the European put option:

Sell 0.4141 futures contracts.
Borrow 122.9537

Question 10.19.

a) The price of a European call option with a strike of 95 is \$24.0058.

b) The price of a European put option with a strike of 95 is \$14.3799.

c) Now, we have for the European call option a premium of \$14.3799 and for the European put option a premium of \$24.0058. Exchanging the four inputs in the formula inverts the call and put relationship. We will encounter a theoretical motivation for this fact in later chapters.

Question 10.20.

a) The price of an American call option with a strike of 95 is \$24.1650.

b) The price of an American put option with a strike of 95 is \$15.2593

c) Now, we have for the American 100-strike call option a premium of \$15.2593 and for the European put option a premium of \$24.165. Both option prices increase as we would have expected, and the relation we observed in question 10.19. continues to hold.

Question 10.21.

Suppose $e^{(r-\delta)h} > u > d$

We short a tailed position of the stock and invest the proceeds at the interest rate: This yields:

	$t = 0$	period h, state $= d$	period h, state $= u$
short stock	$+e^{-\delta h}S$	$-d \times S$	$-u \times S$
loan money	$-e^{-\delta h}S$	$+e^{(r-\delta)h}S$	$+e^{(r-\delta)h}S$
Total	0	> 0	> 0

We have shown that if $e^{(r-\delta)h} > u > d$, there is a true arbitrage possibility.

Conversely, suppose $u > d > e^{(r-\delta)h}$

We then buy a tailed position of the stock and borrow at the prevailing interest rate: This would yield:

	$t = 0$	period h, state $= d$	period h, state $= u$
buy stock	$-e^{-\delta h}S$	$+d \times S$	$+u \times S$
borrow money	$+e^{-\delta h}S$	$-e^{(r-\delta)h}S$	$-e^{(r-\delta)h}S$
Total	0	> 0	> 0

We have shown that if $u > d > e^{(r-\delta)h}$, there is a true arbitrage possibility.