Math 5632 Notes

Black-Scholes Formula

- Review of normal distribution $\mathcal{N}(\mu, \sigma^2)$.
- Black-Scholes Formula

  Assumptions:

  - The return on the stock is normally distributes and independent overtime.
  - The volatility of the return is a constant σ.
  - The dividend yield is a constant δ.
  - The interest rate is a constant r.
  - There is no transaction costs and taxes.

  The payoff of a call option on a stock with strike price K and expiration time T is $V(T) = (S(T) - K)^+$. It can be computed that the price $C_0 = e^{-rT}\mathbb{E}^*((S(T) - K)^+)$ of such a option is given by

  $$C_0 = C(S_0, K, \sigma, r, T, \delta) = S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2)$$

  where N is the CDF of the standard normal distribution, and

  $$d_1 = \frac{\ln(S_0/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln(S_0/K) + (r - \delta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

  Similarly, the price of the corresponding European put option is[1]

  $$P_0 = P(S_0, K, \sigma, r, T, \delta) = K e^{-rT} N(-d_2) - S_0 e^{-\delta T} N(-d_1)$$

- Examples
- Option Greeks

  The following are for call options (C). Put options (P) are similar.

  - Delta: $\Delta = \dfrac{\partial C}{\partial S}$
  - Gamma: $\Gamma = \dfrac{\partial^2 C}{\partial^2 S}$
  - Vega: $\text{Vega} = \dfrac{\partial C}{\partial \sigma}$
  - Psi: $\Psi = \dfrac{\partial C}{\partial \delta}$
  - Theta: $\theta = \dfrac{\partial C(..., T - t, ...)}{\partial t}$
  - Rho: $\rho = \dfrac{\partial C}{\partial r}$

- Exercise: Show that $\Delta = e^{-\delta T} N(d_1)$ for call, and $\Delta = -e^{-\delta T} N(-d_1)$ for put.

[1] If we accept the formula for a call option, then by put-call parity (note that $1 - N(d) = N(-d)$)

$$P = C - Se^{-\delta T} + Ke^{-rT} = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) - Se^{-\delta T} + Ke^{-rT}$$

$$= Ke^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1)$$

- Since $Se^{-\delta T} = F^P_{0,T}(S)$, $Ke^{-rT} = F^P_{0,T}(K)$, and

$$d_1 = \frac{\ln(Se^{-\delta T}/Ke^{-rT}) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln(F^P_{0,T}(S)/F^P_{0,T}(K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

  Black-Scholes formulas can be written as

$$C(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, K) = F^P_{0,T}(S)N(d_1) - F^P_{0,T}(K)N(d_2),$$

$$P(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, K) = F^P_{0,T}(K)N(-d_2) - F^P_{0,T}(S)N(-d_1).$$

- Derivatives are linear $\Rightarrow$ Greeks are linear.
- Option Elasticity $\Omega = \dfrac{\partial C/C}{\partial S/S} = \dfrac{S\partial C}{C\partial S} = \dfrac{S\Delta}{C}$.

  Δ measures dollar risk of an option; Ω measures percentage risk of the option.

- Let α and γ be the expected returns of a stock and an option on it, and r be the (continuous) interest rate. Then

$$\gamma = \frac{\Delta S}{C}\alpha + \left(1 - \frac{\Delta S}{C}\right) r \text{ (why?) } = \Omega\alpha + (1-\Omega)r \Rightarrow \gamma - r = (\alpha - r)\cdot\Omega.$$

  That is the risk premium of an option is the risk premium of the stock times Ω.

- Sharpe Ratio
- Behavior near Maturity
- Implied Volatility