Math 632 Notes

Chapter 20

- Brownian Motion

  A Brownian motion is a stochastic process $Z(t)$ such that:

  - $Z(0) = 0$.
  - $Z(t+s) - Z(t) \sim \mathcal{N}(0, s)$.
  - $Z(t+s_1) - Z(t)$ and $Z(t) - Z(t-s_2)$ are independent.
  - $Z(t)$ has continuous sampling paths.

  Especially, $Z(t) \sim \mathcal{N}(0, t)$.

  $Z(t)$ is a *martingale*: $\mathbb{E}[Z(t+s)|Z(t)] = Z(t)$.

  > The conditional expectation $\mathbb{E}[\,\cdot\,|Z(t)]$ can be viewed as the expected value at time t when everything is known up to time t. So $Z(t)$ or $S(t)$ and so on are treated as constants. Since $Z(t+s) - Z(t) \sim \mathcal{N}(0, s)$ which does not depend on time t, $\mathbb{E}[Z(t+s) - Z(t)|Z(t)] = \mathbb{E}[Z(t+s) - Z(t)] = 0$; since $Z(t)$ is a constant (at time t), $\mathbb{E}[Z(t)|Z(t)] = Z(t)$. We have
  >
  > $$\begin{aligned} \mathbb{E}[Z(t+s)|Z(t)] &= \mathbb{E}[Z(t+s) - Z(t) + Z(t)|Z(t)] \\ &= \mathbb{E}[Z(t+s) - Z(t)|Z(t)] + \mathbb{E}[Z(t)|Z(t)] \\ &= 0 + Z(t) \;=\; Z(t) \end{aligned}$$
  >
  > Note that this is not a proof of $\mathbb{E}[Z(t+s)|Z(t)] = Z(t)$! We did not even define the precise meaning of $\mathbb{E}[Z(t+s)|Z(t)]$.

  EXERCISE. Let $S(t) = S(0)e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma Z(t)}$ be the price of a non-dividend paying stock, where r is the (constant) risk-free interest rate and $Z(t)$ is a Brownian motion. Show that the discounted stock price $e^{-rt}S(t)$ is a martingale.

- Quadratic Variation

  Divide the time interval $[0, T]$ into n equal parts, and write the ith subinterval as $[(i-1)h, ih]$ which has length $h = T/n$.

  Let $\Delta_i Z = Z(ih) - Z((i-1)h)$ be the change of Z over the ith subinterval. Since $\Delta_i Z = Z(ih) - Z((i-1)h) \sim \mathcal{N}(0, h)$, $\Delta_i Z = \sqrt{h} \cdot Z_i$ where the Z_i's are *i.i.d.* standard normal random variables. Then

  $$\sum_{i=1}^{n} (\Delta_i Z)^2 = \sum_{i=1}^{n} h Z_i^2 = \sum_{i=1}^{n} \left(\frac{T}{n}\right) Z_i^2 = T \sum_{i=1}^{n} \frac{Z_i^2}{n}.$$

By the law of large numbers, $\sum_{i=1}^{n} \frac{Z_i^2}{n} \to \mathbb{E}(Z_1^2) = \text{Var}\,(Z_1) = 1$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \sum_{i=1}^{n} [\Delta_i Z]^2 = T,$$

i.e. the quadratic variation $[Z(t), Z(t)]$ of $Z(t)$ over $[0, T]$ is T. This can be written as

$$\int_0^T (dZ(t))^2 = T = \int_0^T dt$$

and this suggests

$$(dZ(t))^2 = dZ(t) \cdot dZ(t) = dt.$$

- Example. Total Variation

  Consider

$$\sum_{i=1}^{n} |\Delta_i Z| = \sum_{i=1}^{n} \sqrt{h}|Z_i| = \sum_{i=1}^{n} \sqrt{\frac{T}{n}}|Z_i| = \sqrt{Tn} \sum_{i=1}^{n} \frac{|Z_i|}{n}.$$

  Since $\sum_{i=1}^{n} \frac{|Z_i|}{n} \to \mathbb{E}(|Z_1|) = \frac{2}{\sqrt{2\pi}}$ (check!) and $\sqrt{Tn} \to \infty$ as $n \to \infty$,

$$\lim_{n\to\infty} \sum_{i=1}^{n} |\Delta_i Z| = \infty,$$

  i.e. the total variation of $Z(t)$ is infinite.

- Exercise. Let $\Delta_i t = ih - (i-1)h = h$ be the change of time over the ith subinterval. Show that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Delta_i Z \cdot \Delta_i t = \lim_{n\to\infty} \sum_{i=1}^{n} [\Delta_i Z] \cdot h = 0, \qquad \lim_{n\to\infty} \sum_{i=1}^{n} \Delta_i t \cdot \Delta_i t = \lim_{n\to\infty} \sum_{i=1}^{n} h^2 = 0$$

  that is, the cross variation of t and $Z(t)$ and the quadratic variation of t on $[0, T]$ are both 0. Therefore, it is reasonable to say that $dZ \cdot dt = 0$ and $dt \cdot dt = 0$.

- Example. (Problem 10 from SOA samples) Consider the Black-Scholes framework. Let $S(t)$ be the stock price at time t, $t \geq 0$. Define $X(t) = \ln[S(t)]$. Which of the following three statements concerning $X(t)$ are true?

  1. $\{X(t), t \geq 0\}$ is an arithmetic Brownian motion.
  2. $\text{Var}[X(t+h) - X(t)] = \sigma^2 h$, $t \geq 0$, $h > 0$.
  3. $\lim_{n\to\infty} \sum_{j=1}^{n} [X(jT/n) - X((j-1)T/n)]^2 = \sigma^2 T$.

Solution. Under the Black-Scholes framework, $S(t) = S(0)e^{(\alpha-\delta)t+\sigma Z(t)}$. So,

$$X(t) = \ln S(0) + (\alpha - \delta)t + \sigma Z(t).$$

We will discuss 1 after Itô's Lemma. Now, let us consider 2 and 3.

$$X(t+h) - X(t) = (\alpha - \delta)h + \sigma[Z(t+h) - Z(t)] \sim \mathcal{N}\left((\alpha-\delta)h, \sigma^2 h\right)$$

So $\mathrm{Var}[X(t+h) - X(t)] = \sigma^2 h$ and 2 is true.

$$\Delta_j X = X(jT/n) - X((j-1)T/n) =$$
$$= (\alpha - \delta)T/n + \sigma[Z(jT/n) - Z((j-1)T/n)] = (\alpha-\delta)\Delta_j t + \sigma \Delta_j Z$$

$$[X(jT/n) - X((j-1)T/n)]^2 = [(\alpha-\delta)\Delta_j t + \sigma\Delta_j Z]^2 =$$
$$= (\alpha-\delta)^2(\Delta_j t)^2 + 2(\alpha-\delta)\sigma\Delta_j t \cdot \Delta_j Z + \sigma^2(\Delta_j Z)^2$$

Then, $\lim_{n\to\infty}\sum_{j=1}^{n}(\Delta_j t)^2 = 0$ and $\lim_{n\to\infty}\sum_{j=1}^{n}\Delta_j t\cdot\Delta_j Z = 0$ imply

$$\lim_{n\to\infty}\sum_{j=1}^{n}[X(jT/n) - X((j-1)T/n)]^2 = \sigma^2\lim_{n\to\infty}\sum_{j=1}^{n}(\Delta_j Z)^2 = \sigma^2 T$$

and so 3 is also true.

- Itô Processes

  An Itô process is a stochastic process $X(t)$ satisfying

  $$dX(t) = \alpha(t, X(t))\,dt + \sigma(t, X(t))\,dZ(t).$$

  - Arithmetic Brownian Motion

    An arithmetic Brownian motion is a $X(t)$ such that

    $$dX(t) = \alpha\,dt + \sigma\,dZ(t)$$

    where both α and σ are constants. X can be written as

    $$X(t) - X(0) = \alpha t + \sigma Z(t).$$

    Since $Z(t) \sim \mathcal{N}(0,t)$, $X(t) - X(0) \sim \mathcal{N}(\alpha t, \sigma^2 t)$.
    $X(t)$ is also called a *drifted* Brownian motion. Its integral form is

    $$X(t) = X(0) + \int_0^t \alpha\,dt + \int_0^t \sigma\,dZ(t).$$

- Geometric Brownian Motion

  A geometic Brownian motion is a $X(t)$ such that

$$dX(t) = \alpha X(t)\,dt + \sigma X(t)\,dZ(t) \quad \text{or} \quad \frac{dX(t)}{X(t)} = \alpha\,dt + \sigma\,dZ(t)$$

  where both α and σ are constants. The solution to the differential equation is

$$X(t) = X(0)e^{\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma Z(t)}$$

  We will verify this after Itô's Lemma. Note that $X(t)$ is a lognormal distribution with

$$\ln X(t) \sim \mathcal{N}\left(\ln X(0) + (\alpha - \sigma^2/2)t, \sigma^2 t\right) \quad \text{and} \quad \mathbb{E}(X(t)) = X(0)e^{\alpha t}.$$

  Under Black-Scholes, the stock price $S(t)$ is a geometric Brownian motion satisfying

$$dS(t) = (\alpha - \delta)S\,dt + \sigma S(t)\,dZ(t).$$

  and so $S(t) = S(0)e^{(\alpha-\delta-\sigma^2/2)t+\sigma Z(t)}$.

- Ornstein-Uhlenbeck Process

  $X(t)$ is a Ornstein-Uhlenbeck process if it satisfies

$$dX(t) = \lambda[\alpha - X(t)]\,dt + \sigma\,dZ(t).$$

  This process has the *mean-reverting* property.

- General Stock Price Process

$$dS(t) = \left[\hat{\alpha}(S(t), t) - \hat{\delta}(S(t), t)\right]\,dt + \hat{\sigma}(S(t), t)\,dZ(t) \qquad (1)$$

  where $\hat{\alpha}$, $\hat{\delta}$, and $\hat{\sigma}$ are instantaneous return, dividend yield, and volatility respectively. If $\hat{\alpha}(S(t), t) = \alpha S(t)$, $\hat{\delta}(S(t), t) = \delta S(t)$, and $\hat{\sigma}(S(t), t) = \sigma S(t)$, then $S(t)$ is the geometric Brownian motion (i.e. lognormal) price in Black-Scholes.

- Multiplication Rule (a.k.a. Box Algebra)

  In the discussion of quadratic variation of $Z(t)$, we get $dZ \cdot dZ = dt$. Similarly, from a discussion of "cross" variation we can get $dZ \cdot dt = 0$, and from a discussion of quadratic variation of t we can get $dt \cdot dt = 0$ (see Exercise in the section above). Sumarising, we have

	dt	$dZ(t)$
dt	0	0
$dZ(t)$	0	dt

Example. For a geometric Brownian motion $X(t)$, we have

$$[dX(t)]^2 = [\alpha X(t)\,dt + \sigma X(t)\,dZ(t)]^2 = \sigma^2 X(t)^2\,dt.$$

Exercise. For a general stock price process, verify that $(dS)^2 = \hat{\sigma}(S(t),t)^2\,dt$.

Example. Find the quadratic variation over $[0,T]$ for an Itô Processes $X(t)$.

- Itô's Lemma

Let X be an Itô process, and $f(x,t)$ be a twice differentiable function. Then

$$df(X,t) = f_x \cdot dX + \frac{1}{2} f_{xx} \cdot (dX)^2 + f_t \cdot dt.$$

Example. Let $S(t)$ be a stock price process such that $dS = (\alpha-\delta)S\,dt + \sigma S\,dZ$, and let $C(S(t),t)$ be the price of an option on S. Then

$$\begin{aligned} dC(S(t),t) &= C_S\,dS + \frac{1}{2}C_{SS}\,(dS)^2 + C_t\,dt \\ &= \left[(\alpha-\delta)SC_S + \frac{1}{2}\sigma^2 S^2 C_{SS} + C_t\right]dt + \sigma S C_S\,dZ \end{aligned}$$

Example. (Geometric BM) Let $Z(t)$ be a Brownian motion. Verify that

$$X(t) = X(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z(t)} \quad \text{satisfies} \quad dX(t) = \mu X(t)\,dt + \sigma X(t)\,dZ(t).$$

Example. (Problem 13 from SOA samples) Let $Z(t)$ be a standard Brownian motion. You are given:

1. $U(t) = 2Z(t) - 2$
2. $V(t) = [Z(t)]^2 - t$
3. $W(t) = t^2 Z(t) - 2\int_0^t sZ(s)\,ds$

Which of the processes defined above has/have zero drift?
(A process X has zero drift if it's differential dX has no dt term.)

Exercise In Chapter 12, the text mentioned a formula (Formula 12.9) for the "volatility" of an option. Explain why that formula is a reasonable definition of "volatility" of an option.

- Real and Risk-Neutral Probability

In Black-Scholes model, stock price is modeled by a geometric Brownian motion:

$$dS = (\alpha-\delta)S\,dt + \sigma S\,dZ(t), \quad \text{or} \quad S(t) = S(0)e^{\left(\alpha-\delta-\frac{\sigma^2}{2}\right)t + \sigma Z(t)}$$

where $Z(t)$ is a Brownian motion under the true (real) probability $\mathbb{P}$, and α is the real expected return. Rewrite the exponent in $S(t)$ as following

$$\begin{aligned}\left(\alpha-\delta-\frac{\sigma^2}{2}\right)t+\sigma Z(t) &= \left(\alpha-\delta-\frac{\sigma^2}{2}\right)t+rt-rt+\sigma Z(t)\\ &= \left(r-\delta-\frac{\sigma^2}{2}\right)t+(\alpha-r)t+\sigma Z(t)\\ &= \left(r-\delta-\frac{\sigma^2}{2}\right)t+\sigma\left[\frac{\alpha-r}{\sigma}t+Z(t)\right]\\ &= \left(r-\delta-\frac{\sigma^2}{2}\right)t+\sigma\tilde{Z}(t)\end{aligned}$$

where

$$\tilde{Z}(t)=\frac{\alpha-r}{\sigma}\cdot t+Z(t).$$

A theorem (Girsanov's Theorem) in probability asserts that *there exists a probability measure* $\mathbb{P}^*$ *under which* $\tilde{Z}(t)$ *is a Brownian motion.* This probability is called the *risk-neutral* probability. In $\tilde{Z}$, the price of the stock S is

$$dS=(r-\delta)S\,dt+\sigma S\,dZ, \quad \text{or} \quad S(t)=S(0)e^{\left(r-\delta-\frac{\sigma^2}{2}\right)t+\sigma\tilde{Z}(t)}.$$

So, under $\mathbb{P}^*$ the *expected* return of the stock is r, the risk-free rate of interest.

Example. (Problem 61 from SOA samples) Assume the Black-Scholes framework. You are given:

1. $S(t)$ is the price of a stock at time t.
2. The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 1%.
3. The stock-price process is given by

$$\frac{dS(t)}{S(t)}=0.05\,dt+0.25\,dZ(t)$$

   where $\{Z(t)\}$ is a standard Brownian motion under the true probability measure.
4. Under the risk-neutral probability measure, the mean of $Z(0.5)$ is -0.03.

Calculate the continuously compounded risk-free interest rate.

Solution. 3 $\Rightarrow$ $\alpha-\delta=0.05$, $\sigma=0.25$ $\Rightarrow$ $\alpha=0.06$. We have

$$\tilde{Z}(t)=\frac{\alpha-r}{\sigma}\cdot t+Z(t) \Rightarrow \tilde{Z}(0.5)=\frac{0.06-r}{0.25}\cdot 0.5+Z(0.5)$$

Since $\mathbb{E}^*[\tilde{Z}(0.5)]=0$ and $\mathbb{E}^*[Z(0.5)]=-0.03$,

$$0=\frac{0.06-r}{0.25}\cdot 0.5-0.03 \Rightarrow r=0.045.$$

EXAMPLE. (Problem 65 from SOA samples) Assume the Black-Scholes framework. You are given:

1. $S(t)$ is the time-t price of a stock, $t \geq 0$.
2. The stock pays dividends continuously at a rate proportional to its price.
3. Under the true probability measure, $\ln[S(2)/S(1)]$ is a normal random variable with mean 0.10.
4. Under the risk-neutral probability measure, $\ln[S(5)/S(3)]$ is a normal random variable with mean 0.06.
5. The continuously compounded risk-free interest rate is 4%.
6. The time-0 price of a European put option on the stock is 10.
7. For delta-hedging at time 0 one unit of the put option with shares of the stock, the cost of stock shares is 20.

Calculate the absolute value of the time-0 continuously compounded expected rate of return on the put option.

Solution. (We want to find α_P. Recall that $\alpha_P - r = \Omega_P(\alpha - r)$.)

From 3, $\alpha - \delta - \frac{\sigma^2}{2} = 0.1$ and from 4, $\left(r - \delta - \frac{\sigma^2}{2}\right) \cdot 2 = 0.06$ or $r - \delta - \frac{\sigma^2}{2} = 0.03 \;\Rightarrow$

$$\alpha - r = \left(\alpha - \delta - \frac{\sigma^2}{2}\right) - \left(r - \delta - \frac{\sigma^2}{2}\right) = 0.1 - 0.03 = 0.07.$$

$6 \;\Rightarrow\; P = 10$ and $7 \;\Rightarrow\; \Delta_P S = -20$. So, $\Omega_P = -20/10 = -2$.

$$\alpha_P = \Omega_P(\alpha - r) + r = (-2) \cdot (0.07) + 0.04 = -0.10$$

EXERCISE. (Problem 23 from SOA samples)

- Risk-Neutral Pricing

Let $V(T)$ be the payoff at time $t = T$ of a contingent claim V (i.e. option) on a stock S. Then the (time 0) price of the claim is

$$V(0) = e^{-rT}\mathbb{E}^*[V(T)]$$

where $\mathbb{E}^*$ is the expectation under the risk-neutral probability $\mathbb{P}^*$.

EXERCISE. For a call option, check that the price given by $C = e^{-rT}\mathbb{E}^*[(S(T) - K)^+]$ is the same as that given by the Black-Scholes formula.

In general, for any $t < T$, the price of V at time t is

$$V(t) = e^{-r(T-t)}\mathbb{E}^*[V(T)|S(t)].$$

That is, the price of V is the discounted conditional risk-neutral expectation of $V(T)$.

When it is clear within the context (for example, the stock price is given in terms of r instead of α), we drop the star in $\mathbb{E}^*$ by just writing $\mathbb{E}$.

- Claims on S^a

  Let $S(t)$ be the price process following $dS = (r-\delta)S\,dt + \sigma S\,dZ$ as in BS. Then $S(t) = S(0)e^{(r-\delta-\frac{\sigma^2}{2})t+\sigma Z(t)}$. Consider an option that pays $V(T) = S(T)^a$ at time T. We want to find the time 0 price of this option, or find the prepaid forward price of $S(T)^a$. $S^a = S(0)^a e^{a(r-\delta-\frac{\sigma^2}{2})t+a\sigma Z(t)} \Rightarrow$

$$\mathbb{E}(S(T)^a) = S(0)^a e^{a(r-\delta-\frac{\sigma^2}{2})T+\frac{a^2\sigma^2 T}{2}} = S(0)^a e^{[a(r-\delta)+\frac{1}{2}a(a-1)\sigma^2]T} \Rightarrow$$

$$\Rightarrow F^P_{0,T}(S^a) = e^{-rT}S(0)^a e^{[a(r-\delta)+\frac{1}{2}a(a-1)\sigma^2]T}$$

$$F_{0,T}(S^a) = S(0)^a e^{[a(r-\delta)+\frac{1}{2}a(a-1)\sigma^2]T}$$

  More generally for $0 < t < T$, $S(T) = S(t)e^{\left(r-\delta-\frac{\sigma^2}{2}\right)(T-t)+\sigma(Z(T)-Z(t))}$, and this implies

$$F^P_{t,T}(S^a) = e^{-r(T-t)}S(t)^a e^{[a(r-\delta)+\frac{1}{2}a(a-1)\sigma^2](T-t)}.$$

  Example. (Problem 16 from SOA samples) Assume that the Black-Scholes framework holds. Let $S(t)$ be the price of a nondividend-paying stock at time t, $t \geq 0$. The stock's volatility is 20%, and the continuously compounded risk-free interest rate is 4%. You are interested in contingent claims with payoff being the stock price raised to some power. For $0 \leq t < T$, consider the equation $F^P_{t,T}[S(T)^x] = S(t)^x$, where the left-hand side is the prepaid forward price at time t of a contingent claim that pays $S(T)^x$ at time T. A solution for the equation is $x = 1$. Determine another x that solves the equation.

  Solution.

$$F_{t,T}(S^x) = e^{-r(T-t)}S(t)^x e^{[x(r-\delta)+\frac{1}{2}x(x-1)\sigma^2](T-t)} = S(t)^x$$

$$\Rightarrow \quad -r(T-t) + [x(r-\delta) + \frac{1}{2}x(x-1)\sigma^2](T-t) = 0$$

$$\Rightarrow \quad \cdots \quad \Rightarrow x^2 + x - 2 = 0 \quad \Rightarrow \quad x = 1, \quad x = -2.$$

- Sharpe Ratio

  Let the expected return and volatility of an asset are α and σ respectively. The Sharpe ratio of the asset is $\frac{\alpha-r}{\sigma}$, the risk premium divided by the volatility. Note that Sharpe ratio is the rate of *shift* in the *change* of variable from real market uncertainty Z to risk-neutral uncertainty $\tilde{Z}$: $\tilde{Z}(t) = \frac{\alpha-r}{\sigma}\cdot t + Z(t)$. Therefore, Sharpe ratio is also called the *market price of the risk.*

  If two *tradable* assets were driven by the same BM:

$$dS_1 = \alpha_1 S_1\,dt + \sigma_1 S_1\,dZ, \quad dS_2 = \alpha_2 S_2\,dt + \sigma_2 S_2\,dZ$$

  then their Sharpe ratios are equal: $\dfrac{\alpha_1 - r}{\sigma_1} = \dfrac{\alpha_2 - r}{\sigma_2}$.

If not, there would be an arbitrage opportunity. Say $\frac{\alpha_1 - r}{\sigma_1} > \frac{\alpha_2 - r}{\sigma_2}$. Build a portfolio I that longs $\frac{1}{\sigma_1 S_1}$ share of asset 1 and shorts $\frac{1}{\sigma_2 S_2}$ share of asset 2 and finance the portfolio by risk-free bond (i.e. borrowing/saving in risk-free money market). Then the change in the value of the portfolio is

$$\begin{aligned} dI &= \left(\frac{1}{\sigma_1 S_1}\, dS_1 - \frac{r}{\sigma_1}\, dt\right) - \left(\frac{1}{\sigma_2 S_2}\, dS_2 - \frac{r}{\sigma_2}\, dt\right) \\ &= \left(\frac{\alpha_1 - r}{\sigma_1}\right) dt - \left(\frac{\alpha_2 - r}{\sigma_2}\right) dt \; > \; 0 \end{aligned}$$

an arbitrage.

Example. (Problem 12 from SOA samples) Consider two non-dividend paying assets X and Y. There is a single source of uncertainty which is captured by a standard Brownian motion $\{Z(t)\}$. The prices of the assets satisfy the stochastic differential equations

$$\frac{dX(t)}{X(t)} = 0.07\, dt + 0.12\, dZ(t), \qquad \frac{dY(t)}{Y(t)} = A\, dt + B\, dZ(t)$$

where A and B are constants. You are also given:

1. $d[\ln Y(t)] = \mu\, dt + 0.085\, dZ(t)$;
2. The continuously compounded risk-free interest rate is 4%.

Determine A.

Solution. First note that $\dfrac{A - r}{B} = \dfrac{0.07 - r}{0.12}$, the Sharpe ratios are equal.

$$Y(t) = Y(0)e^{\left(A - \frac{B^2}{2}\right)t + B\cdot Z(t)} \;\Rightarrow\; \ln Y(t) = \ln Y(0) + \left(A - \frac{B^2}{2}\right) t + B \cdot Z(t)$$

This last equation and condition 1 imply $B = 0.085$ (we also get $A - \frac{B^2}{2} = \mu$, but we don't need it here). Then

$$A = \frac{0.07 - r}{0.12} \cdot B + r = 0.06125.$$

Example. (Problem 66 from SOA samples)

Example. (Problem 67 from SOA samples)