Exam MFE/3F

Sample Questions and Solutions

#1 to #76

August 18, 2010

1. Consider a European call option and a European put option on a nondividend-paying stock. You are given:

(i)   The current price of the stock is 60.

(ii)   The call option currently sells for 0.15 more than the put option.

(iii)   Both the call option and put option will expire in 4 years.

(iv)   Both the call option and put option have a strike price of 70.

Calculate the continuously compounded risk-free interest rate.

(A) 0.039

(B) 0.049

(C) 0.059

(D) 0.069

(E) 0.079

Solution to (1)   Answer: (A)

The put-call parity formula (for a European call and a European put on a stock with the same strike price and maturity date) is

$$C - P = F_{0,T}^{P}(S) - F_{0,T}^{P}(K)$$

$$= F_{0,T}^{P}(S) - PV_{0,T}(K)$$

$$= F_{0,T}^{P}(S) - Ke^{-rT}$$

$$= S_0 - Ke^{-rT},$$

because the stock pays no dividends

We are given that $C - P = 0.15$, $S_0 = 60$, $K = 70$ and $T = 4$. Then, $r = 0.039$.

Remark 1: If the stock pays n dividends of fixed amounts $D_1, D_2,\ldots, D_n$ at fixed times t_1, $t_2,\ldots, t_n$ prior to the option maturity date, T, then the put-call parity formula for European put and call options is

$$C - P = F_{0,T}^{P}(S) - Ke^{-rT}$$
$$= S_0 - PV_{0,T}(\text{Div}) - Ke^{-rT},$$

where $PV_{0,T}(\text{Div}) = \sum_{i=1}^{n} D_i e^{-rt_i}$ is the present value of all dividends up to time T. The

difference, $S_0 - PV_{0,T}(\text{Div})$, is the *prepaid forward price* $F_{0,T}^{P}(S)$.

Remark 2: The put-call parity formula above does not hold for American put and call options. For the American case, the parity relationship becomes

$$S_0 - PV_{0,T}(\text{Div}) - K \leq C - P \leq S_0 - Ke^{-rT}.$$

This result is given in Appendix 9A of McDonald (2006) but is not required for Exam MFE/3F. Nevertheless, you may want to try proving the inequalities as follows:
For the first inequality, consider a portfolio consisting of a European call plus an amount of cash equal to $PV_{0,T}(\text{Div}) + K$.
For the second inequality, consider a portfolio of an American put option plus one share of the stock.

2. Near market closing time on a given day, you lose access to stock prices, but some European call and put prices for a stock are available as follows:

Strike Price	Call Price	Put Price
$40	$11	$3
$50	$6	$8
$55	$3	$11

All six options have the same expiration date.

After reviewing the information above, John tells Mary and Peter that no arbitrage opportunities can arise from these prices.

Mary disagrees with John. She argues that one could use the following portfolio to obtain arbitrage profit: Long one call option with strike price 40; short three call options with strike price 50; lend $1; and long some calls with strike price 55.

Peter also disagrees with John. He claims that the following portfolio, which is different from Mary's, can produce arbitrage profit: Long 2 calls and short 2 puts with strike price 55; long 1 call and short 1 put with strike price 40; lend $2; and short some calls and long the same number of puts with strike price 50.

Which of the following statements is true?

(A) Only John is correct.

(B) Only Mary is correct.

(C) Only Peter is correct.

(D) Both Mary and Peter are correct.

(E) None of them is correct.

Solution to (2)   Answer: (D)

The prices are not arbitrage-free. To show that Mary's portfolio yields arbitrage profit, we follow the analysis in Table 9.7 on page 302 of McDonald (2006).

	Time 0	Time T			
		$S_T < 40$	$40 \leq S_T < 50$	$50 \leq S_T < 55$	$S_T \geq 55$
Buy 1 call Strike 40	-11	0	$S_T - 40$	$S_T - 40$	$S_T - 40$
Sell 3 calls Strike 50	$+18$	0	0	$-3(S_T - 50)$	$-3(S_T - 50)$
Lend $1	-1	e^{rT}	e^{rT}	e^{rT}	e^{rT}
Buy 2 calls strike 55	-6	0	0	0	$2(S_T - 55)$
Total	0	$e^{rT} > 0$	$e^{rT} + S_T - 40$ > 0	$e^{rT} + 2(55 - S_T)$ > 0	$e^{rT} > 0$

Peter's portfolio makes arbitrage profit, because:

	Time-0 cash flow	Time-T cash flow
Buy 2 calls & sells 2 puts Strike 55	$2(-3 + 11) = 16$	$2(S_T - 55)$
Buy 1 call & sell 1 put Strike 40	$-11 + 3 = -8$	$S_T - 40$
Lend $2	-2	$2e^{rT}$
Sell 3 calls & buy 3 puts Strike 50	$3(6 - 8) = -6$	$3(50 - S_T)$
Total	0	$2e^{rT}$

Remarks: Note that Mary's portfolio has no put options. The call option prices are not arbitrage-free; they do not satisfy the convexity condition (9.17) on page 300 of McDonald (2006). The time-T cash flow column in Peter's portfolio is due to the identity
$$\max[0, S - K] - \max[0, K - S] = S - K$$
(see also page 44).

In *Loss Models*, the textbook for Exam C/4, $\max[0, \alpha]$ is denoted as α_+. It appears in the context of stop-loss insurance, $(S - d)_+$, with S being the claim random variable and d the deductible. The identity above is a particular case of
$$x = x_+ - (-x)_+,$$
which says that every number is the difference between its positive part and negative part.

3. An insurance company sells single premium deferred annuity contracts with return linked to a stock index, the time-t value of one unit of which is denoted by $S(t)$. The contracts offer a minimum guarantee return rate of $g\%$. At time 0, a single premium of amount π is paid by the policyholder, and $\pi \times y\%$ is deducted by the insurance company. Thus, at the contract maturity date, T, the insurance company will pay the policyholder

$$\pi \times (1 - y\%) \times \text{Max}[S(T)/S(0), (1 + g\%)^T].$$

You are given the following information:

(i) The contract will mature in one year.

(ii) The minimum guarantee rate of return, $g\%$, is 3%.

(iii) Dividends are incorporated in the stock index. That is, the stock index is constructed with all stock dividends reinvested.

(iv) $S(0) = 100$.

(v) The price of a one-year European put option, with strike price of $103, on the stock index is $15.21.

Determine $y\%$, so that the insurance company does not make or lose money on this contract.

Solution to (3)

The payoff at the contract maturity date is

$$\pi \times (1 - y\%) \times \text{Max}[S(T)/S(0), (1 + g\%)^T]$$
$$= \pi \times (1 - y\%) \times \text{Max}[S(1)/S(0), (1 + g\%)^1] \quad \text{because } T = 1$$
$$= [\pi/S(0)](1 - y\%)\text{Max}[S(1), S(0)(1 + g\%)]$$
$$= (\pi/100)(1 - y\%)\text{Max}[S(1), 103] \quad \text{because } g = 3 \ \& \ S(0)=100$$
$$= (\pi/100)(1 - y\%)\{S(1) + \text{Max}[0, 103 - S(1)]\}.$$

Now, $\text{Max}[0, 103 - S(1)]$ is the payoff of a one-year European put option, with strike price \$103, on the stock index; the time-0 price of this option is given to be is \$15.21. Dividends are incorporated in the stock index (i.e., $\delta = 0$); therefore, $S(0)$ is the time-0 price for a time-1 payoff of amount $S(1)$. Because of the no-arbitrage principle, the time-0 price of the contract must be

$$(\pi/100)(1 - y\%)\{S(0) + 15.21\}$$
$$= (\pi/100)(1 - y\%) \times 115.21.$$

Therefore, the "break-even" equation is

$$\pi = (\pi/100)(1 - y\%) \times 115.21,$$

or

$$y\% = 100 \times (1 - 1/1.1521)\% = 13.202\%$$

Remarks:
(i)   Many stock indexes, such as S&P500, do not incorporate dividend reinvestments. In such cases, the time-0 cost for receiving $S(1)$ at time 1 is the prepaid forward price $F_{0,1}^P(S)$, which is less than $S(0)$.

(ii) The identities

$$\text{Max}[S(T), K] = K + \text{Max}[S(T) - K, 0] = K + (S(T) - K)_+$$

and

$$\text{Max}[S(T), K] = S(T) + \text{Max}[0, K - S(T)] = S(T) + (K - S(T))_+$$

can lead to a derivation of the put-call parity formula. Such identities are useful for understanding Section 14.6 *Exchange Options* in McDonald (2006).

4. For a two-period binomial model, you are given:

(i)   Each period is one year.

(ii) The current price for a nondividend-paying stock is 20.

(iii) $u = 1.2840$, where u is one plus the rate of capital gain on the stock per period if the stock price goes up.

(iv) $d = 0.8607$, where d is one plus the rate of capital loss on the stock per period if the stock price goes down.

(v)   The continuously compounded risk-free interest rate is 5%.

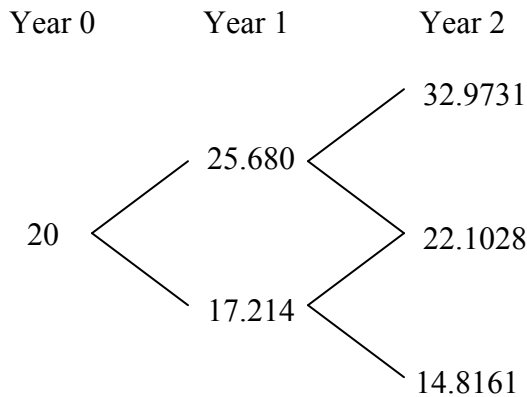Calculate the price of an American call option on the stock with a strike price of 22.

(A)   0

(B)   1

(C)   2

(D)   3

(E)   4

Solution to (4)   Answer: (C)

First, we construct the two-period binomial tree for the stock price.

Year 0   Year 1   Year 2

20

25.680

17.214

32.9731

22.1028

14.8161

The calculations for the stock prices at various nodes are as follows:

$S_u = 20 \times 1.2840 = 25.680$
$S_d = 20 \times 0.8607 = 17.214$
$S_{uu} = 25.68 \times 1.2840 = 32.9731$
$S_{ud} = S_{du} = 17.214 \times 1.2840 = 22.1028$
$S_{dd} = 17.214 \times 0.8607 = 14.8161$

The risk-neutral probability for the stock price to go up is
$$p^* = \frac{e^{rh} - d}{u - d} = \frac{e^{0.05} - 0.8607}{1.2840 - 0.8607} = 0.4502.$$
Thus, the risk-neutral probability for the stock price to go down is 0.5498.

If the option is exercised at time 2, the value of the call would be
$C_{uu} = (32.9731 - 22)_+ = 10.9731$
$C_{ud} = (22.1028 - 22)_+ = 0.1028$
$C_{dd} = (14.8161 - 22)_+ = 0$

If the option is European, then $C_u = e^{-0.05}[0.4502C_{uu} + 0.5498C_{ud}] = 4.7530$ and
$C_d = e^{-0.05}[0.4502C_{ud} + 0.5498C_{dd}] = 0.0440$.
But since the option is American, we should compare C_u and C_d with the value of the
option if it is exercised at time 1, which is 3.68 and 0, respectively. Since $3.68 < 4.7530$
and $0 < 0.0440$, it is not optimal to exercise the option at time 1 whether the stock is in
the up or down state. Thus the value of the option at time 1 is either 4.7530 or 0.0440.

Finally, the value of the call is
$C = e^{-0.05}[0.4502(4.7530) + 0.5498(0.0440)] = 2.0585$.

Remark: Since the stock pays no dividends, the price of an American call is the same as that of a European call. See pages 294-295 of McDonald (2006). The European option price can be calculated using the binomial probability formula. See formula (11.17) on page 358 and formula (19.1) on page 618 of McDonald (2006). The option price is

$$e^{-r(2h)}[\binom{2}{2}p*^2 C_{uu} + \binom{2}{1}p*(1-p*)C_{ud} + \binom{2}{0}(1-p*)^2 C_{dd}]$$

$= e^{-0.1}[(0.4502)^2 \times 10.9731 + 2 \times 0.4502 \times 0.5498 \times 0.1028 + 0]$

$= 2.0507$

5. Consider a 9-month dollar-denominated American put option on British pounds. You are given that:

(i)   The current exchange rate is 1.43 US dollars per pound.

(ii) The strike price of the put is 1.56 US dollars per pound.

(iii) The volatility of the exchange rate is $\sigma = 0.3$.

(iv) The US dollar continuously compounded risk-free interest rate is 8%.

(v)   The British pound continuously compounded risk-free interest rate is 9%.

Using a three-period binomial model, calculate the price of the put.

Solution to (5)

Each period is of length $h = 0.25$. Using the first two formulas on page 332 of McDonald (2006):
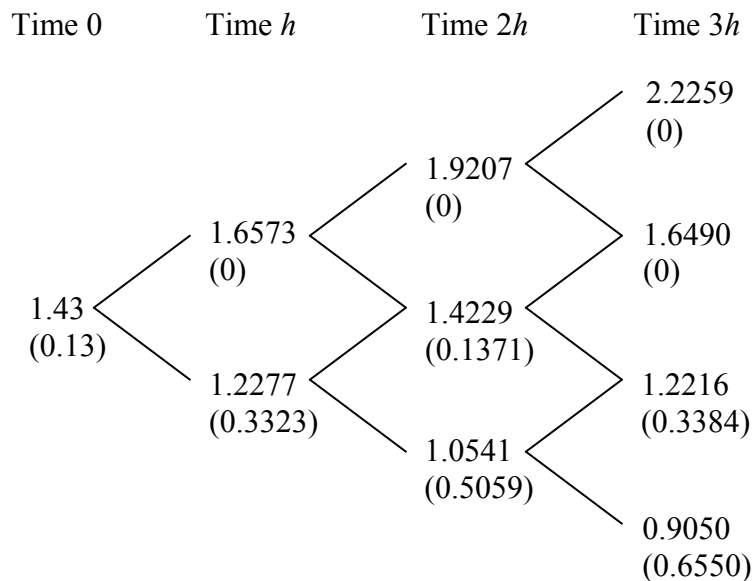
$$u = \exp[-0.01 \times 0.25 + 0.3 \times \sqrt{0.25}\,] = \exp(0.1475) = 1.158933,$$
$$d = \exp[-0.01 \times 0.25 - 0.3 \times \sqrt{0.25}\,] = \exp(-0.1525) = 0.858559.$$

Using formula (10.13), the risk-neutral probability of an up move is

$$p^* = \frac{e^{-0.01 \times 0.25} - 0.858559}{1.158933 - 0.858559} = 0.4626.$$

The risk-neutral probability of a down move is thus 0.5374. The 3-period binomial tree for the exchange rate is shown below. The numbers within parentheses are the payoffs of the put option if exercised.

Time 0	Time h	Time $2h$	Time $3h$

```
Time 0        Time h        Time 2h       Time 3h

                                            2.2259
                                            (0)
                              1.9207
                              (0)
                1.6573                      1.6490
                (0)                         (0)
    1.43                      1.4229
    (0.13)                    (0.1371)
                1.2277                      1.2216
                (0.3323)                    (0.3384)
                              1.0541
                              (0.5059)
                                            0.9050
                                            (0.6550)
```

The payoffs of the put at maturity (at time $3h$) are
$P_{uuu} = 0$, $P_{uud} = 0$, $P_{udd} = 0.3384$ and $P_{ddd} = 0.6550$.

Now we calculate values of the put at time $2h$ for various states of the exchange rate.

If the put is European, then
$P_{uu} = 0$,
$P_{ud} = e^{-0.02}[0.4626 P_{uud} + 0.5374 P_{udd}] = 0.1783$,
$P_{dd} = e^{-0.02}[0.4626 P_{udd} + 0.5374 P_{ddd}] = 0.4985$.
But since the option is American, we should compare P_{uu}, P_{ud} and P_{dd} with the values of the option if it is exercised at time $2h$, which are 0, 0.1371 and 0.5059, respectively. Since $0.4985 < 0.5059$, it is optimal to exercise the option at time $2h$ if the exchange rate has gone down two times before. Thus the values of the option at time $2h$ are $P_{uu} = 0$, $P_{ud} = 0.1783$ and $P_{dd} = 0.5059$.

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Now we calculate values of the put at time h for various states of the exchange rate.

If the put is European, then
$P_u = e^{-0.02}[0.4626P_{uu} + 0.5374P_{ud}] = 0.0939,$
$P_d = e^{-0.02}[0.4626P_{ud} + 0.5374P_{dd}] = 0.3474.$
But since the option is American, we should compare P_u and P_d with the values of the option if it is exercised at time h, which are 0 and 0.3323, respectively. Since 0.3474 > 0.3323, it is not optimal to exercise the option at time h. Thus the values of the option at time h are $P_u = 0.0939$ and $P_d = 0.3474$.

Finally, discount and average P_u and P_d to get the time-0 price,

$$P = e^{-0.02}[0.4626P_u + 0.5374P_d] = 0.2256.$$

Since it is greater than 0.13, it is not optimal to exercise the option at time 0 and hence the price of the put is 0.2256.

Remarks:

(i)   Because $\dfrac{e^{(r-\delta)h} - e^{(r-\delta)h-\sigma\sqrt{h}}}{e^{(r-\delta)h+\sigma\sqrt{h}} - e^{(r-\delta)h-\sigma\sqrt{h}}} = \dfrac{1-e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} = \dfrac{1}{1+e^{\sigma\sqrt{h}}}$, we can also

calculate the risk-neutral probability p^* as follows:
$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.3\sqrt{0.25}}} = \frac{1}{1+e^{0.15}} = 0.46257.$$

(ii) $$1 - p^* = 1 - \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{e^{\sigma\sqrt{h}}}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{-\sigma\sqrt{h}}}.$$

(iii)   Because $\sigma > 0$, we have the inequalities

$$p^* < \tfrac{1}{2} < 1 - p^*.$$

6. You are considering the purchase of 100 units of a 3-month 25-strike European call option on a stock.

You are given:

(i) The Black-Scholes framework holds.

(ii) The stock is currently selling for 20.

(iii) The stock's volatility is 24%.

(iv) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.

(v) The continuously compounded risk-free interest rate is 5%.

Calculate the price of the block of 100 options.

(A) 0.04

(B) 1.93

(C) 3.50

(D) 4.20

(E) 5.09

Solution to (6)   Answer: (C)

$$C(S,K,\sigma,r,T,\delta) = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2) \qquad (12.1)$$
with
$$d_1 = \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \qquad (12.2a)$$
$$d_2 = d_1 - \sigma\sqrt{T} \qquad (12.2b)$$

Because $S = \$20$, $K = \$25$, $\sigma = 0.24$, $r = 0.05$, $T = 3/12 = 0.25$, and $\delta = 0.03$, we have
$$d_1 = \frac{\ln(20/25) + (0.05 - 0.03 + \frac{1}{2}0.24^2)0.25}{0.24\sqrt{0.25}} = -1.75786$$
and
$$d_2 = -1.75786 - 0.24\sqrt{0.25} = -1.87786$$

Because d_1 and d_2 are negative, use $N(d_1) = 1 - N(-d_1)$ and $N(d_2) = 1 - N(-d_2)$.
In Exam MFE/3F, round $-d_1$ to 1.76 before looking up the standard normal distribution
table. Thus, $N(d_1)$ is $1 - 0.9608 = 0.0392$. Similarly, round $-d_2$ to 1.88, and $N(d_2)$ is thus
$1 - 0.9699 = 0.0301$.

Formula (12.1) becomes
$$C = 20e^{-(0.03)(0.25)}(0.0392) - 25e^{-(0.05)(0.25)}(0.0301) = 0.0350$$

Cost of the block of 100 options $= 100 \times 0.0350 = \$3.50$.

7.   Company A is a U.S. international company, and Company B is a Japanese local company. Company A is negotiating with Company B to sell its operation in Tokyo to Company B. The deal will be settled in Japanese yen. To avoid a loss at the time when the deal is closed due to a sudden devaluation of yen relative to dollar, Company A has decided to buy at-the-money dollar-denominated yen put of the European type to hedge this risk.

You are given the following information:

(i)   The deal will be closed 3 months from now.

(ii)   The sale price of the Tokyo operation has been settled at 120 billion Japanese yen.

(iii)   The continuously compounded risk-free interest rate in the U.S. is 3.5%.

(iv)   The continuously compounded risk-free interest rate in Japan is 1.5%.

(v)   The current exchange rate is 1 U.S. dollar = 120 Japanese yen.

(vi)   The natural logarithm of the yen per dollar exchange rate is an arithmetic Brownian motion with daily volatility 0.261712%.

(vii)   1 year = 365 days; 3 months = ¼ year.

Calculate Company A's option cost.

Solution to (7)

Let $X(t)$ be the exchange rate of U.S. dollar per Japanese yen at time t. That is, at time t,
$$¥1 = \$X(t).$$
We are given that $X(0) = 1/120$.

At time ¼, Company A will receive ¥ 120 billion, which is exchanged to
$[120 billion × $X(¼)$]$. However, Company A would like to have
$$\$ \text{Max}[1 \text{ billion}, \ 120 \text{ billion} \times X(¼)],$$
which can be decomposed as
$$\$120 \text{ billion} \times X(¼) \ + \ \$ \text{Max}[1 \text{ billion} \ - \ 120 \text{ billion} \times X(¼), 0],$$
or
$$\$120 \text{ billion} \times \{X(¼) \ + \ \text{Max}[120^{-1} - X(¼), 0]\}.$$

Thus, Company A purchases 120 billion units of a put option whose payoff three months from now is
$$\$ \text{Max}[120^{-1} - X(¼), 0].$$

The exchange rate can be viewed as the price, in US dollar, of a traded asset, which is the Japanese yen. The continuously compounded risk-free interest rate in Japan can be interpreted as δ, the dividend yield of the asset. See also page 381 of McDonald (2006) for the *Garman-Kohlhagen model*. Then, we have
$$r = 0.035, \delta = 0.015, S = X(0) = 1/120, K = 1/120, T = ¼.$$

Because the logarithm of the exchange rate of yen per dollar is an arithmetic Brownian motion, its negative, which is the logarithm of the exchange rate of dollar per yen, is also an arithmetic Brownian motion and has the SAME volatility. Therefore, $\{X(t)\}$ is a geometric Brownian motion, and the put option can be priced using the Black-Scholes formula for European put options. It remains to determine the value of σ, which is given by the equation
$$\sigma\sqrt{\frac{1}{365}} = 0.261712 \text{ %}.$$
Hence,
$$\sigma = 0.05.$$
Therefore,
$$d_1 = \frac{(r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{(0.035 - 0.015 + 0.05^2/2)/4}{0.05\sqrt{1/4}} = 0.2125$$
and
$$d_2 = d_1 - \sigma\sqrt{T} = 0.2125 - 0.05/2 = 0.1875.$$
By (12.3) of McDonald (2006), the time-0 price of 120 billion units of the put option is
$$\$120 \text{ billion} \times [Ke^{-rT}N(-d_2) - X(0)e^{-\delta T}N(-d_1)]$$
$$= \$ [e^{-rT}N(-d_2) - e^{-\delta T}N(-d_1)] \text{ billion} \qquad \text{because } K = X(0) = 1/120$$
$$= \$ \{e^{-rT}[1 - N(d_2)] - e^{-\delta T}[1 - N(d_1)]\} \text{ billion}$$

In Exam MFE/3F, you will be given a standard normal distribution table. Use the value of $N(0.21)$ for $N(d_1)$, and $N(0.19)$ for $N(d_1)$.

Because $N(0.21) = 0.5832$, $N(0.19) = 0.5753$, $e^{-rT} = e^{-0.035 \times 0.25} = 0.9963$, and $e^{-\delta T} = e^{-0.015 \times 0.25} = 0.9963$, Company A's option cost is
$$0.9963 \times 0.4247 - 0.9963 \times 0.4168 \ = \ 0.005747 \text{ billion} \ \approx \ \$5.75 \text{ million}.$$

Remarks:
(i) Suppose that the problem is to be solved using options on the exchange rate of Japanese yen per US dollar, i.e., using yen-denominated options. Let
$$\$1 \ = \ ¥U(t)$$
at time t, i.e., $U(t) = 1/X(t)$.

Because Company A is worried that the dollar may increase in value with respect to the yen, it buys 1 billion units of a 3-month yen-denominated European *call* option, with exercise price ¥120. The payoff of the option at time ¼ is
$$¥ \operatorname{Max}[U(¼) \ - \ 120, 0].$$

To apply the Black-Scholes call option formula (12.1) to determine the time-0 price in yen, use
$$r = 0.015, \ \delta = 0.035, \ S = U(0) = 120, \ K = 120, \ T = ¼, \text{ and } \sigma = 0.05.$$
Then, divide this price by 120 to get the time-0 option price in dollars. We get the same price as above, because d_1 here is $-d_2$ of above.

The above is a special case of formula (9.7) on page 292 of McDonald (2006).

(ii) There is a cheaper solution for Company A. At time 0, borrow
$$¥ \ 120 \times \exp(- \ ¼ \ r_¥) \text{ billion,}$$
and immediately convert this amount to US dollars. The loan is repaid with interest at time ¼ when the deal is closed.
On the other hand, with the option purchase, Company A will benefit if the yen increases in value with respect to the dollar.

8. You are considering the purchase of a 3-month 41.5-strike American call option on a nondividend-paying stock.

You are given:
(i) The Black-Scholes framework holds.
(ii) The stock is currently selling for 40.
(iii) The stock's volatility is 30%.
(iv) The current call option delta is 0.5.

Determine the current price of the option.

(A) $20 - 20.453 \int_{-\infty}^{0.15} e^{-x^2/2} dx$

(B) $20 - 16.138 \int_{-\infty}^{0.15} e^{-x^2/2} dx$

(C) $20 - 40.453 \int_{-\infty}^{0.15} e^{-x^2/2} dx$

(D) $16.138 \int_{-\infty}^{0.15} e^{-x^2/2} dx - 20.453$

(E) $40.453 \int_{-\infty}^{0.15} e^{-x^2/2} dx - 20.453$

Solution to (8)   Answer: (D)

Since it is never optimal to exercise an American call option before maturity if the stock pays no dividends, we can price the call option using the European call option formula

$$C = SN(d_1) - Ke^{-rT}N(d_2),$$

where $d_1 = \dfrac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

Because the call option delta is $N(d_1)$ and it is given to be 0.5, we have $d_1 = 0$. Hence,

$$d_2 = -0.3 \times \sqrt{0.25} = -0.15 .$$

To find the continuously compounded risk-free interest rate, use the equation

$$d_1 = \frac{\ln(40/41.5) + (r + \frac{1}{2} \times 0.3^2) \times 0.25}{0.3\sqrt{0.25}} = 0,$$

which gives $r = 0.1023$.

Thus,
$$C = 40N(0) - 41.5e^{-0.1023 \times 0.25}N(-0.15)$$
$$= 20 - 40.453[1 - N(0.15)]$$
$$= 40.453N(0.15) - 20.453$$
$$= \frac{40.453}{\sqrt{2\pi}} \int_{-\infty}^{0.15} e^{-x^2/2}dx - 20.453$$
$$= 16.138 \int_{-\infty}^{0.15} e^{-x^2/2}dx - 20.453$$

9.   Consider the Black-Scholes framework. A market-maker, who delta-hedges, sells a three-month at-the-money European call option on a nondividend-paying stock.

You are given:

(i)   The continuously compounded risk-free interest rate is 10%.

(ii)   The current stock price is 50.

(iii)   The current call option delta is 0.6179.

(iv)   There are 365 days in the year.

If, after one day, the market-maker has zero profit or loss, determine the stock price move over the day.

(A)   0.41

(B)   0.52

(C)   0.63

(D)   0.75

(E)   1.11

Solution to (9)

According to the first paragraph on page 429 of McDonald (2006), such a stock price move is given by plus or minus of
$$\sigma\, S(0)\, \sqrt{h}\, ,$$
where $h = 1/365$ and $S(0) = 50$. It remains to find σ.

Because the stock pays no dividends (i.e., $\delta = 0$), it follows from the bottom of page 383 that $\Delta = N(d_1)$. By the condition $N(d_1) = 0.6179$, we get $d_1 = 0.3$. Because $S = K$ and $\delta = 0$, formula (12.2a) is
$$d_1 \;=\; \frac{(r + \sigma^2/2)T}{\sigma\sqrt{T}}$$
or
$$\tfrac{1}{2}\sigma^2 \;-\; \frac{d_1}{\sqrt{T}}\,\sigma \;+\; r \;=\; 0.$$
With $d_1 = 0.3$, $r = 0.1$, and $T = 1/4$, the quadratic equation becomes
$$\tfrac{1}{2}\sigma^2 \;-\; 0.6\sigma \;+\; 0.1 \;=\; 0,$$
whose roots can be found by using the quadratic formula or by factorization,
$$\tfrac{1}{2}(\sigma - 1)(\sigma - 0.2) \;=\; 0.$$
We reject $\sigma = 1$ because such a volatility seems too large (and none of the five answers fit). Hence,
$$\sigma\, S(0)\, \sqrt{h} \;=\; 0.2 \times 50 \times 0.052342 \;\approx\; 0.52.$$

Remarks: The Itô's Lemma in Chapter 20 of McDonald (2006) can help us understand Section 13.4. Let $C(S, t)$ be the price of the call option at time t if the stock price is S at that time. We use the following notation
$$C_S(S,\,t) \;=\; \frac{\partial}{\partial S} C(S,t)\,, \quad C_{SS}(S,\,t) \;=\; \frac{\partial^2}{\partial S^2} C(S,t)\,, \quad C_t(S,\,t) \;=\; \frac{\partial}{\partial t} C(S,t)\,,$$
$$\Delta_t \;=\; C_S(S(t),\, t)\,, \quad \Gamma_t \;=\; C_{SS}(S(t),\, t)\,, \quad \theta_t \;=\; C_t(S(t),\, t).$$

At time t, the so-called market-maker sells one call option, and he delta-hedges, i.e., he buys delta, Δ_t, shares of the stock. At time $t + dt$, the stock price moves to $S(t + dt)$, and option price becomes $C(S(t + dt),\, t + dt)$. The interest expense for his position is
$$[\Delta_t S(t) \,-\, C(S(t),\, t)](r\, dt).$$
Thus, his profit at time $t + dt$ is
$$\Delta_t[S(t + dt) - S(t)] \;-\; [C(S(t + dt),\, t + dt) - C(S(t),\, t)] \;-\; [\Delta_t S(t) - C(S(t),\, t)](r\, dt)$$
$$=\; \Delta_t\, dS(t) \;-\; dC(S(t),\, t) \;-\; [\Delta_t S(t) - C(S(t),\, t)](r\, dt). \qquad\qquad (*)$$

We learn from Section 20.6 that
$$dC(S(t),\, t) \;=\; C_S(S(t),\, t)dS(t) \;+\; \tfrac{1}{2}C_{SS}(S(t),\, t)[dS(t)]^2 \;+\; C_t(S(t),\, t)dt \qquad (20.28)$$
$$=\; \Delta_t\, dS(t) \;+\; \tfrac{1}{2}\Gamma_t[dS(t)]^2 \;+\; \theta_t\, dt. \qquad\qquad (**)$$

Because $dS(t) = S(t)[\alpha\, dt + \sigma\, dZ(t)]$, it follows from the multiplication rules (20.17) that
$$[dS(t)]^2 = [S(t)]^2\, \sigma^2\, dt, \qquad\qquad (***)$$
which should be compared with (13.8). Substituting $(***)$ in $(**)$ yields
$$dC(S(t), t) = \Delta_t\, dS(t) + \tfrac{1}{2}\Gamma_t\,[S(t)]^2\, \sigma^2\, dt + \theta_t\, dt,$$
application of which to $(*)$ shows that the market-maker's profit at time $t + dt$ is
$$-\{\tfrac{1}{2}\Gamma_t\,[S(t)]^2\, \sigma^2\, dt + \theta_t\, dt\} - [\Delta_t S(t) - C(S(t), t)](r\,dt)$$
$$= -\{\tfrac{1}{2}\Gamma_t\,[S(t)]^2\, \sigma^2 + \theta_t + [\Delta_t S(t) - C(S(t), t)]r\}dt, \qquad (****)$$
which is the same as (13.9) if dt can be h.

Now, at time t, the value of stock price, $S(t)$, is known. Hence, expression $(****)$, the market-maker's profit at time $t+dt$, is not stochastic. If there are no riskless arbitrages, then quantity within the braces in $(****)$ must be zero,
$$C_t(S, t) + \tfrac{1}{2}\sigma^2 S^2 C_{SS}(S, t) + rS C_S(S, t) - rC(S, t) = 0,$$
which is the celebrated Black-Scholes equation (13.10) for the price of an option on a nondividend-paying stock. Equation (21.11) in McDonald (2006) generalizes (13.10) to the case where the stock pays dividends continuously and proportional to its price.

Let us consider the substitutions
$$dt \to h$$
$$dS(t) = S(t + dt) - S(t) \to S(t + h) - S(t),$$
$$dC(S(t), t) = C(S(t + dt), t + dt) - C(S(t), t) \to C(S(t + h), t + h) - C(S(t), t).$$
Then, equation $(**)$ leads to the approximation formula
$$C(S(t + h), t + h) - C(S(t), t) \approx \Delta_t\,[S(t + h) - S(t)] + \tfrac{1}{2}\Gamma_t\,[S(t + h) - S(t)]^2 + \theta_t\, h,$$
which is given near the top of page 665. Figure 13.3 on page 426 is an illustration of this approximation. Note that in formula (13.6) on page 426, the equal sign, $=$, should be replaced by an approximately equal sign such as $\approx$.

Although $(***)$ holds because $\{S(t)\}$ is a geometric Brownian motion, the analogous equation,
$$[S(t + h) - S(t)]^2 = [\sigma S(t)]^2 h, \qquad h > 0,$$
which should be compared with (13.8) on page 429, almost never holds. If it turns out that it holds, then the market maker's profit is approximated by the right-hand side of (13.9). The expression is zero because of the Black-Scholes partial differential equation.

10. Consider the Black-Scholes framework. Let $S(t)$ be the stock price at time t, $t \geq 0$. Define $X(t) = \ln[S(t)]$.

You are given the following three statements concerning $X(t)$.

(i) $\{X(t), t \geq 0\}$ is an arithmetic Brownian motion.

(ii) $\mathrm{Var}[X(t+h) - X(t)] = \sigma^2 h, \qquad t \geq 0, h > 0.$

(iii) $\displaystyle \lim_{n \to \infty} \sum_{j=1}^{n} [X(jT/n) - X((j-1)T/n)]^2 = \sigma^2 T.$

(A) Only (i) is true

(B) Only (ii) is true

(C) Only (i) and (ii) are true

(D) Only (i) and (iii) are true

(E) (i), (ii) and (iii) are true

Solution to (10)   Answer: (E)

(i) is true. That $\{S(t)\}$ is a geometric Brownian motion means exactly that its logarithm is an arithmetic Brownian motion. (Also see the solution to problem (11).)

(ii) is true. Because $\{X(t)\}$ is an arithmetic Brownian motion, the increment, $X(t + h) - X(t)$, is a normal random variable with variance $\sigma^2 h$. This result can also be found at the bottom of page 605.

(iii) is true. Because $X(t) = \ln S(t)$, we have
$$X(t + h) - X(t) = \mu h + \sigma[Z(t + h) - Z(t)],$$
where $\{Z(t)\}$ is a (standard) Brownian motion and $\mu = \alpha - \delta - \frac{1}{2}\sigma^2$. (Here, we assume the stock price follows (20.25), but the actual value of μ is not important.) Then,
$$[X(t + h) - X(t)]^2 = \mu^2 h^2 + 2\mu h\sigma[Z(t + h) - Z(t)] + \sigma^2[Z(t + h) - Z(t)]^2.$$
With $h = T/n$,

$$\sum_{j=1}^{n}[X(jT/n) - X((j-1)T/n)]^2$$

$$= \mu^2 T^2/n + 2\mu(T/n)\,\sigma[Z(T) - Z(0)] + \sigma^2\sum_{j=1}^{n}[Z(jT/n) - Z((j-1)T/n)]^2.$$

As $n \to \infty$, the first two terms on the last line become 0, and the sum becomes T according to formula (20.6) on page 653.

Remarks: What is called "arithmetic Brownian motion" is the textbook is called "Brownian motion" by many other authors. What is called "Brownian motion" is the textbook is called "standard Brownian motion" by others.

Statement (iii) is a non-trivial result: The limit of sums of stochastic terms turns out to be deterministic. A consequence is that, if we can observe the prices of a stock over a time interval, no matter how short the interval is, we can determine the value of σ by evaluating the quadratic variation of the natural logarithm of the stock prices. Of course, this is under the assumption that the stock price follows a geometric Brownian

motion. This result is a reason why the true stock price process (20.25) and the risk-neutral stock price process (20.26) must have the same σ. A discussion on realized quadratic variation can be found on page 755 of McDonald (2006).

A quick "proof" of the quadratic variation formula (20.6) can be obtained using the multiplication rule (20.17c). The left-hand side of (20.6) can be seen as $\int_0^T [dZ(t)]^2$.

Formula (20.17c) states that $[dZ(t)]^2 = dt$. Thus,

$$\int_0^T [dZ(t)]^2 \; = \; \int_0^T dt \; = \; T.$$

11. Consider the Black-Scholes framework. You are given the following three statements on variances, conditional on knowing $S(t)$, the stock price at time t.

(i) $\mathrm{Var}[\ln S(t + h) \mid S(t)] = \sigma^2 h, \qquad h > 0.$

(ii) $\mathrm{Var}\left[\dfrac{dS(t)}{S(t)} \middle| S(t)\right] = \sigma^2 \, dt$

(iii) $\mathrm{Var}[S(t + dt) \mid S(t)] = S(t)^2 \, \sigma^2 \, dt$

(A)   Only (i) is true

(B)   Only (ii) is true

(C)   Only (i) and (ii) are true

(D)   Only (ii) and (iii) are true

(E)   (i), (ii) and (iii) are true

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Here are some facts about geometric Brownian motion. The solution of the stochastic differential equation

$$\frac{dS(t)}{S(t)} = \alpha\, dt + \sigma\, dZ(t) \qquad\qquad (20.1)$$

is

$$S(t) = S(0)\exp[(\alpha - \tfrac{1}{2}\sigma^2)t + \sigma Z(t)]. \qquad\qquad (*)$$

Formula (*), which can be verified to satisfy (20.1) by using Itô's Lemma, is equivalent to formula (20.29), which is the solution of the stochastic differential equation (20.25). It follows from (*) that

$$S(t+h) = S(t)\exp[(\alpha - \tfrac{1}{2}\sigma^2)h + \sigma[Z(t+h) - Z(t)]], \quad h \geq 0. \qquad (**)$$

From page 650, we know that the random variable $[Z(t+h) - Z(t)]$ has the same distribution as $Z(h)$, i.e., it is normal with mean 0 and variance h.

Solution to (11)   Answer: (E)

(i) is true: The logarithm of equation (**) shows that given the value of $S(t)$, $\ln[S(t+h)]$ is a normal random variable with mean $(\ln[S(t)] + (\alpha - \tfrac{1}{2}\sigma^2)h)$ and variance $\sigma^2 h$. See also the top paragraph on page 650 of McDonald (2006).

(ii) is true: $\mathrm{Var}\left[\dfrac{dS(t)}{S(t)}\middle|S(t)\right] = \mathrm{Var}[\alpha\, dt + \sigma\, dZ(t)|S(t)]$

$$= \mathrm{Var}[\alpha\, dt + \sigma\, dZ(t)|Z(t)],$$

because it follows from (*) that knowing the value of $S(t)$ is equivalent to knowing the value of $Z(t)$. Now,

$$\mathrm{Var}[\alpha dt + \sigma\, dZ(t)|Z(t)] = \mathrm{Var}[\sigma\, dZ(t)|Z(t)]$$
$$= \sigma^2\, \mathrm{Var}[dZ(t)|Z(t)]$$
$$= \sigma^2\, \mathrm{Var}[dZ(t)] \qquad \because \text{ independent increments}$$
$$= \sigma^2\, dt.$$

Remark: The unconditional variance also has the same answer: $\mathrm{Var}\left[\dfrac{dS(t)}{S(t)}\right] = \sigma^2\, dt.$

(iii) is true because (ii) is the same as

$$\text{Var}[dS(t) \mid S(t)] = S(t)^2 \, \sigma^2 \, dt,$$

and

$$\begin{aligned}
\text{Var}[dS(t) \mid S(t)] &= \text{Var}[S(t + dt) - S(t) \mid S(t)] \\
&= \text{Var}[S(t + dt) \mid S(t)].
\end{aligned}$$

A direct derivation for (iii):

$$\begin{aligned}
\text{Var}[S(t + dt) \mid S(t)] &= \text{Var}[S(t + dt) - S(t) \mid S(t)] \\
&= \text{Var}[dS(t) \mid S(t)] \\
&= \text{Var}[\alpha \, S(t)dt + \sigma \, S(t)dZ(t) \mid S(t)] \\
&= \text{Var}[\sigma \, S(t) \, dZ(t) \mid S(t)] \\
&= [\sigma \, S(t)]^2 \, \text{Var}[dZ(t) \mid S(t)] \\
&= [\sigma \, S(t)]^2 \, \text{Var}[Z(t + dt) - Z(t) \mid S(t)] \\
&= [\sigma \, S(t)]^2 \, \text{Var}[Z(dt)] \\
&= [\sigma \, S(t)]^2 \, dt
\end{aligned}$$

We can also show that (iii) is true by means of the formula for the variance of a lognormal random variable (McDonald 2006, eq. 18.14): It follows from formula (**) on the last page that

$$\begin{aligned}
\text{Var}[S(t + h) \mid S(t)] &= \text{Var}[S(t) \exp[(\alpha - \tfrac{1}{2}\sigma^2)h + \sigma[Z(t + h) - Z(t)]] \mid S(t)] \\
&= [S(t)]^2 \, \exp[2(\alpha - \tfrac{1}{2}\sigma^2)h] \, \text{Var}[\exp[\sigma[Z(t + h) - Z(t)]] \mid S(t)] \\
&= [S(t)]^2 \, \exp[2(\alpha - \tfrac{1}{2}\sigma^2)h] \, \text{Var}[\exp[\sigma Z(h)]]] \\
&= [S(t)]^2 \, \exp[2(\alpha - \tfrac{1}{2}\sigma^2)h] \, e^{h\sigma^2}(e^{h\sigma^2} - 1) \\
&= [S(t)]^2 \, \exp[2(\alpha - \tfrac{1}{2}\sigma^2)h] \, e^{h\sigma^2}(h\sigma^2 + \cdots).
\end{aligned}$$

Thus,

$$\text{Var}[S(t + dt) \mid S(t)] = [S(t)]^2 \times 1 \times 1 \times (dt \times \sigma^2),$$

which is (iii).

12. Consider two nondividend-paying assets X and Y. There is a single source of uncertainty which is captured by a standard Brownian motion $\{Z(t)\}$. The prices of the assets satisfy the stochastic differential equations

$$\frac{dX(t)}{X(t)} = 0.07dt + 0.12dZ(t)$$

and

$$\frac{dY(t)}{Y(t)} = Adt + BdZ(t),$$

where A and B are constants.

You are also given:

(i) $d[\ln Y(t)] = \mu dt + 0.085 dZ(t)$;

(ii) The continuously compounded risk-free interest rate is 4%.

Determine A.

(A) 0.0604

(B) 0.0613

(C) 0.0650

(D) 0.0700

(E) 0.0954

Solution to (12)   Answer: (B)

If $f(x)$ is a twice-differentiable function of one variable, then Itô's Lemma (page 664) simplifies as

$$\mathrm{d}f(Y(t)) = f'(Y(t))\mathrm{d}Y(t) + \tfrac{1}{2}f''(Y(t))[\mathrm{d}Y(t)]^2,$$

because $\dfrac{\partial}{\partial t}f(x) = 0.$

If $f(x) = \ln x$, then $f'(x) = 1/x$ and $f''(x) = -1/x^2$. Hence,

$$\mathrm{d}[\ln Y(t)] = \frac{1}{Y(t)}\mathrm{d}Y(t) + \frac{1}{2}\left(-\frac{1}{[Y(t)]^2}\right)[\mathrm{d}Y(t)]^2. \qquad (1)$$

We are given that

$$\mathrm{d}Y(t) = Y(t)[A\mathrm{d}t + B\mathrm{d}Z(t)]. \qquad (2)$$

Thus,

$$[\mathrm{d}Y(t)]^2 = \{Y(t)[A\mathrm{d}t + B\mathrm{d}Z(t)]\}^2 = [Y(t)]^2 B^2\,\mathrm{d}t, \qquad (3)$$

by applying the multiplication rules (20.17) on pages 658 and 659. Substituting (2) and (3) in (1) and simplifying yields

$$\mathrm{d}\,[\ln Y(t)] = (A - \frac{B^2}{2})\mathrm{d}t + B\mathrm{d}Z(t).$$

It thus follows from condition (i) that $B = 0.085$.

It is pointed out in Section 20.4 that two perfectly positively correlated assets must have the same Sharpe ratio. Thus,

$$(0.07 - 0.04)/0.12 = (A - 0.04)/B = (A - 0.04)/0.085$$

Therefore, $A = 0.04 + 0.085(0.25) = 0.06125 \approx 0.0613$

13. Let $\{Z(t)\}$ be a standard Brownian motion. You are given:

(i) $U(t) = 2Z(t) - 2$

(ii) $V(t) = [Z(t)]^2 - t$

(iii) $W(t) = t^2 Z(t) - 2\int_0^t sZ(s)\mathrm{d}s$

Which of the processes defined above has / have zero drift?

(A) $\{V(t)\}$ only

(B) $\{W(t)\}$ only

(C) $\{U(t)\}$ and $\{V(t)\}$ only

(D) $\{V(t)\}$ and $\{W(t)\}$ only

(E) All three processes have zero drift.

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Solution to (13)   Answer: (E)

Apply Itô's Lemma.

(i) $dU(t) = 2dZ(t) - 0 = 0dt + 2dZ(t)$.

Thus, the stochastic process $\{U(t)\}$ has zero drift.

(ii) $dV(t) = d[Z(t)]^2 - dt$.

$$d[Z(t)]^2 = 2Z(t)dZ(t) + \frac{2}{2}[dZ(t)]^2$$

$$= 2Z(t)dZ(t) + dt$$

by the multiplication rule (20.17c) on page 659. Thus,

$$dV(t) = 2Z(t)dZ(t).$$

The stochastic process $\{V(t)\}$ has zero drift.

(iii) $dW(t) = d[t^2 Z(t)] - 2t\, Z(t)dt$

Because

$$d[t^2 Z(t)] = t^2 dZ(t) + 2tZ(t)dt,$$

we have

$$dW(t) = t^2 dZ(t).$$

The process $\{W(t)\}$ has zero drift.

14. You are using the Vasicek one-factor interest-rate model with the short-rate process calibrated as

$$dr(t) = 0.6[b - r(t)]dt + \sigma dZ(t).$$

For $t \leq T,$ let $P(r, t, T)$ be the price at time t of a zero-coupon bond that pays \$1 at time $T,$ if the short-rate at time t is $r.$ The price of each zero-coupon bond in the Vasicek model follows an Itô process,

$$\frac{dP[r(t),t,T]}{P[r(t),t,T]} = \alpha[r(t), t, T]\, dt - q[r(t), t, T]\, dZ(t), \qquad t \leq T.$$

You are given that $\alpha(0.04, 0, 2) = 0.04139761.$

Find $\alpha(0.05, 1, 4).$

Solution to (14)

For $t < T$, $\alpha(r, t, T)$ is the time-t continuously compounded expected rate of return on the zero-coupon bond that matures for 1 at time T, with the short-rate at time t being r.

Because all bond prices are driven by a single source of uncertainties, $\{Z(t)\}$, the no-arbitrage condition implies that the ratio, $\dfrac{\alpha(r,t,T)-r}{q(r,t,T)}$, does not depend on T. See (24.16) on page 782 and (20.24) on page 660 of McDonald (2006).

In the Vasicek model, the ratio is set to be ϕ, a constant. Thus, we have

$$\frac{\alpha(0.05, 1, 4) - 0.05}{q(0.05, 1, 4)} = \frac{\alpha(0.04, 0, 2) - 0.04}{q(0.04, 0, 2)}. \qquad (*)$$

To finish the problem, we need to know q, which is the coefficient of $-dZ(t)$ in $\dfrac{dP[r(t),t,T]}{P[r(t),t,T]}$. To evaluate the numerator, we apply Itô's Lemma:

$$dP[r(t), t, T] \quad = \quad P_t[r(t), t, T]dt + P_r[r(t), t, T]dr(t) + \tfrac{1}{2}P_{rr}[r(t), t, T][dr(t)]^2,$$

which is a portion of (20.10). Because $dr(t) = a[b - r(t)]dt + \sigma dZ(t)$, we have $[dr(t)]^2 = \sigma^2 dt$, which has no dZ term. Thus, we see that

$$q(r, t, T) \;=\; -\sigma P_r(r, t, T)/P(r, t, T) \qquad\qquad \text{which is a special case of (24.12)}$$

$$=\; -\sigma\frac{\partial}{\partial r}\ln[P(r, t, T)].$$

In the Vasicek model and in the Cox-Ingersoll-Ross model, the zero-coupon bond price is of the form

$$P(r, t, T) \;=\; A(t, T)\, e^{-B(t,\,T)r};$$

hence,

$$q(r, t, T) \;=\; -\sigma\frac{\partial}{\partial r}\ln[P(r, t, T)] \;=\; \sigma B(t, T).$$

In fact, both $A(t, T)$ and $B(t, T)$ are functions of the time to maturity, $T - t$. In the Vasicek model, $B(t, T) = [1 - e^{-a(T-t)}]/a$. Thus, equation (*) becomes

$$\frac{\alpha(0.05, 1, 4) - 0.05}{1 - e^{-a(4-1)}} = \frac{\alpha(0.04, 0, 2) - 0.04}{1 - e^{-a(2-0)}}.$$

Because $a = 0.6$ and $\alpha(0.04, 0, 2) = 0.04139761$, we get $\alpha(0.05, 1, 4) = 0.05167$.

Remarks:

(i) The second equation in the problem is equation (24.1) [or (24.13)] of MacDonald (2006). In its first printing, the minus sign on the right-hand side is a plus sign.

(ii) Unfortunately, zero-coupon bond prices are denoted as $P(r, t, T)$ and also as $P(t, T, r)$ in McDonald (2006).

(iii) One can remember the formula,

$$B(t, T) \ = \ [1 - e^{-a(T-t)}]/a,$$

in the Vasicek model as $\bar{a}_{\overline{T-t}|\text{force of interest}\,=\,a}$, the present value of a continuous

annuity-certain of rate 1, payable for $T - t$ years, and evaluated at force of interest a,

where a is the "speed of mean reversion" for the associated short-rate process.

(iv) If the zero-coupon bond prices are of the so-called *affine form*,
$$P(r, t, T) = A(t, T)\, e^{-B(t,\,T)r},$$
where $A(t, T)$ and $B(t, T)$ are independent of r, then (24.12) becomes
$$q(r, t, T) \ = \ \sigma(r)B(t, T).$$
Thus, (24.17) is
$$\phi(r, t) \ = \ \frac{\alpha(r,t,T)-r}{q(r,t,T)} \ = \ \frac{\alpha(r,t,T)-r}{\sigma(r)B(t,T)},$$
from which we obtain
$$\alpha(r, t, T) \ = \ r \ + \ \phi(r, t)\sigma(r)\, B(t, T).$$
In the Vasicek model, $\sigma(r) = \sigma$, $\phi(r, t) = \phi$, and
$$\alpha(r, t, T) \ = \ r \ + \ \phi\sigma B(t, T).$$
In the CIR model, $\sigma(r) = \sigma\sqrt{r}$, $\phi(r, t) = \dfrac{\bar{\phi}\sqrt{r}}{\sigma}$, and
$$\alpha(r, t, T) \ = \ r \ + \ \bar{\phi}rB(t,T).$$

In either model, $A(t, T)$ and $B(t, T)$ depend on the variables t and T by means of their difference $T - t$, which is the time to maturity.

(v) Formula (24.20) on page 783 of McDonald (2006) is

$$P(r, t, T) \ = \ E^*[\exp(-\int_t^T r(s)\,ds) \mid r(t) = r],$$

where the asterisk signifies that the expectation is taken with respect to the risk-neutral probability measure. Under the risk-neutral probability measure, the expected rate of return on each asset is the risk-free interest rate. Now, (24.13) is

$$\frac{dP[r(t),t,T]}{P[r(t),t,T]} = \alpha[r(t), t, T]\,dt - q[r(t), t, T]\,dZ(t)$$

$$= r(t)\,dt - q[r(t), t, T]\,dZ(t) + \{\alpha[r(t), t, T] - r(t)\}dt$$

$$= r(t)\,dt - q[r(t), t, T]\Big\{dZ(t) - \frac{\alpha[r(t),t,T] - r(t)}{q[r(t),t,T]}\,dt\Big\}$$

$$= r(t)\,dt - q[r(t), t, T]\{dZ(t) - \phi[r(t), t]dt\}. \qquad (**)$$

Let us define the stochastic process $\{\tilde{Z}(t)\}$ by

$$\tilde{Z}(t) = Z(t) - \int_0^t \phi[r(s), s]ds.$$

Then, applying

$$\tilde{Z}(t) = dZ(t) - \phi[r(t), t]dt \qquad\qquad (***)$$

to (**) yields

$$\frac{dP[r(t),t,T]}{P[r(t),t,T]} = r(t)dt - q[r(t), t, T]d\tilde{Z}(t),$$

which is analogous to (20.26) on page 661. The risk-neutral probability measure is such that $\{\tilde{Z}(t)\}$ is a standard Brownian motion.

Applying (***) to equation (24.2) yields

$$dr(t) = a[r(t)]dt + \sigma[r(t)]dZ(t)$$

$$= a[r(t)]dt + \sigma[r(t)]\{d\tilde{Z}(t) + \phi[r(t), t]dt\}$$

$$= \{a[r(t)] + \sigma[r(t)]\phi[r(t), t]\}dt + \sigma[r(t)]d\tilde{Z}(t),$$

which is (24.19) on page 783 of McDonald (2006).

15. You are given the following incomplete Black-Derman-Toy interest rate tree model for the effective annual interest rates:
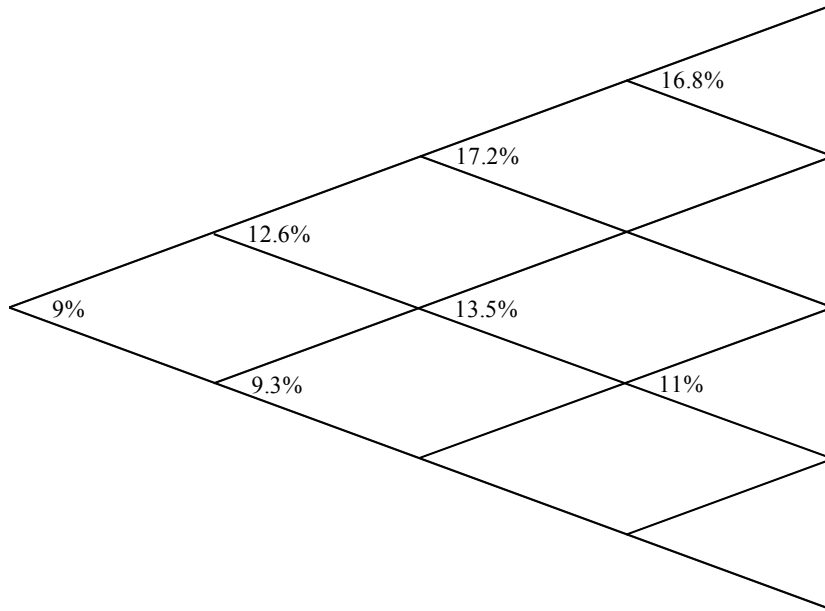
Calculate the price of a year-4 caplet for the notional amount of $100. The cap rate is 10.5%.

Solution to (15)

First, let us fill in the three missing interest rates in the B-D-T binomial tree. In terms of the notation in Figure 24.4 of McDonald (2006), the missing interest rates are r_d, r_{ddd}, and r_{uud}. We can find these interest rates, because in each period, the interest rates in different states are terms of a *geometric progression*.

$$\frac{0.135}{r_{dd}} = \frac{0.172}{0.135} \Rightarrow r_{dd} = 10.6\%$$

$$\frac{r_{uud}}{0.11} = \frac{0.168}{r_{uud}} \Rightarrow r_{uud} = 13.6\%$$

$$\left(\frac{0.11}{r_{ddd}}\right)^2 = \frac{0.168}{0.11} \Rightarrow r_{ddd} = 8.9\%$$

The payment of a year-4 caplet is made at year 4 (time 4), and we consider its discounted value at year 3 (time 3). At year 3 (time 3), the binomial model has four nodes; at that time, a year-4 caplet has one of four values:

$$\frac{16.8 - 10.5}{1.168} = 5.394, \; \frac{13.6 - 10.5}{1.136} = 2.729, \; \frac{11 - 10.5}{1.11} = 0.450, \text{ and } 0 \text{ because } r_{ddd} = 8.9\%$$

which is less than 10.5%.

For the Black-Derman-Toy model, the risk-neutral probability for an up move is ½. We now calculate the caplet's value in each of the three nodes at time 2:

$$\frac{(5.394 + 2.729)/2}{1.172} = 3.4654, \; \frac{(2.729 + 0.450)/2}{1.135} = 1.4004, \; \frac{(0.450 + 0)/2}{1.106} = 0.2034.$$

Then, we calculate the caplet's value in each of the two nodes at time 1:

$$\frac{(3.4654 + 1.4004)/2}{1.126} = 2.1607, \; \frac{(1.40044 + 0.2034)/2}{1.093} = 0.7337.$$

Finally, the time-0 price of the year-4 caplet is $\dfrac{(2.1607 + 0.7337)/2}{1.09} = 1.3277$.

Remarks:

(i) The discussion on caps and caplets on page 805 of McDonald (2006) involves a loan. This is not necessary.

(ii) If your copy of McDonald was printed before 2008, then you need to correct the typographical errors on page 805; see
http://www.kellogg.northwestern.edu/faculty/mcdonald/htm/typos2e_01.html

(iii) In the earlier version of this problem, we mistakenly used the term "year-3 caplet" for "year-4 caplet."

Alternative Solution: The payoff of the year-4 caplet is made at year 4 (at time 4). In a binomial lattice, there are 16 paths from time 0 to time 4.

For the *uuuu* path, the payoff is $(16.8 - 10.5)_+$

For the *uuud* path, the payoff is also $(16.8 - 10.5)_+$

For the *uudu* path, the payoff is $(13.6 - 10.5)_+$

For the *uudd* path, the payoff is also $(13.6 - 10.5)_+$

:
:

We discount these payoffs by the one-period interest rates (annual interest rates) along interest-rate paths, and then calculate their average with respect to the risk-neutral probabilities. In the Black-Derman-Toy model, the risk-neutral probability for each interest-rate path is the same. Thus, the time-0 price of the caplet is

$$
\frac{1}{16} \left\{ \frac{(16.8-10.5)_+}{1.09\times1.126\times1.172\times1.168} + \frac{(16.8-10.5)_+}{1.09\times1.126\times1.172\times1.168} \right.
$$

$$
\left. + \frac{(13.6-10.5)_+}{1.09\times1.126\times1.172\times1.136} + \frac{(13.6-10.5)_+}{1.09\times1.126\times1.172\times1.136} + \ldots\ldots\ldots\ldots \right\}
$$

$$
= \frac{1}{8} \left\{ \frac{(16.8-10.5)_+}{1.09\times1.126\times1.172\times1.168} \right.
$$

$$
+ \frac{(13.6-10.5)_+}{1.09\times1.126\times1.172\times1.136} + \frac{(13.6-10.5)_+}{1.09\times1.126\times1.135\times1.136} + \frac{(13.6-10.5)_+}{1.09\times1.093\times1.135\times1.136}
$$

$$
+ \frac{(11-10.5)_+}{1.09\times1.126\times1.135\times1.11} + \frac{(11-10.5)_+}{1.09\times1.093\times1.135\times1.11} + \frac{(11-10.5)_+}{1.09\times1.093\times1.106\times1.11}
$$

$$
\left. + \frac{(9-10.5)_+}{1.09\times1.093\times1.106\times1.09} \right\} = 1.326829.
$$

Remark: In this problem, the payoffs are path-independent. The "backward induction" method in the earlier solution is more efficient. However, if the payoffs are path-dependent, then the price will need to be calculated by the "path-by-path" method illustrated in this alternative solution.

16. Assume that the Black-Scholes framework holds. Let $S(t)$ be the price of a nondividend-paying stock at time t, $t \geq 0$. The stock's volatility is 20%, and the continuously compounded risk-free interest rate is 4%.

You are interested in contingent claims with payoff being the stock price raised to some power. For $0 \leq t < T$, consider the equation

$$F_{t,T}^P[S(T)^x] = S(t)^x ,$$

where the left-hand side is the prepaid forward price at time t of a contingent claim that pays $S(T)^x$ at time T. A solution for the equation is $x = 1$.

Determine another x that solves the equation.

(A) −4
(B) −2
(C) −1
(D) 2
(E) 4

Solution to (16)   Answer (B)

It follows from (20.30) in Proposition 20.3 that
$$F_{t,T}^P[S(T)^x] = S(t)^x \exp\{[-r + x(r-\delta) + \tfrac{1}{2}x(x-1)\sigma^2](T-t)\},$$
which equals $S(t)^x$ if and only if
$$-r + x(r-\delta) + \tfrac{1}{2}x(x-1)\sigma^2 = 0.$$
This is a quadratic equation of x. With $\delta = 0$, the quadratic equation becomes
$$0 = -r + xr + \tfrac{1}{2}x(x-1)\sigma^2$$
$$= (x-1)(\tfrac{1}{2}\sigma^2 x + r).$$
Thus, the solutions are 1 and $-2r/\sigma^2 = -2(4\%)/(20\%)^2 = -2$, which is (B).

Remarks:

(i) McDonald (2006, Section 20.7) has provided three derivations for (20.30). Here is another derivation. Define $Y = \ln[S(T)/S(t)]$. Then,

$$F_{t,T}^P[S(T)^x] = E_t^*[e^{-r(T-t)} S(T)^x] \qquad \because \text{ Prepaid forward price}$$

$$= E_t^*[e^{-r(T-t)} (S(t)e^Y)^x] \qquad \because \text{ Definition of } Y$$

$$= e^{-r(T-t)} S(t)^x E_t^*[e^{xY}]. \qquad \because \text{ The value of } S(t) \text{ is not random at time } t$$

The problem is to find x such that $e^{-r(T-t)} E_t^*[e^{xY}] = 1$. The expectation $E_t^*[e^{xY}]$ is the *moment-generating function* of the random variable Y at the value x. Under the risk-neutral probability measure, Y is normal with mean $(r - \delta - \tfrac{1}{2}\sigma^2)(T-t)$ and variance $\sigma^2(T-t)$. Thus, by the moment-generating function formula for a normal random variable,
$$E_t^*[e^{xY}] = \exp[x(r-\delta-\tfrac{1}{2}\sigma^2)(T-t) + \tfrac{1}{2}x^2\sigma^2(T-t)],$$
and the problem becomes finding x such that
$$0 = -r(T-t) + x(r-\delta-\tfrac{1}{2}\sigma^2)(T-t) + \tfrac{1}{2}x^2\sigma^2(T-t),$$
which is the same quadratic equation as above.

(ii) Applying the quadratic formula, one finds that the two solutions of
$$-r + x(r-\delta) + \tfrac{1}{2}x(x-1)\sigma^2 = 0$$
are $x = h_1$ and $x = h_2$ of Section 12.6 in McDonald (2006). A reason for this "coincidence" is that $x = h_1$ and $x = h_2$ are the values for which the stochastic process $\{e^{-rt} S(t)^x\}$ becomes a *martingale*. Martingales are mentioned on page 651 of McDonald (2006).

(iii) Before time T, the contingent claim does not pay anything. Thus, the prepaid forward price at time t is in fact the time-t price of the contingent claim.

17. You are to estimate a nondividend-paying stock's annualized volatility using its prices in the past nine months.

Month	Stock Price ($/share)
1	80
2	64
3	80
4	64
5	80
6	100
7	80
8	64
9	80

Calculate the historical volatility for this stock over the period.

(A) 83%

(B) 77%

(C) 24%

(D) 22%

(E) 20%

Solution to (17)   Answer (A)

This problem is based on Sections 11.3 and 11.4 of McDonald (2006), in particular, Table 11.1 on page 361.

Let $\{r_j\}$ denote the continuously compounded monthly returns. Thus, $r_1 = \ln(80/64)$, $r_2 = \ln(64/80)$, $r_3 = \ln(80/64)$, $r_4 = \ln(64/80)$, $r_5 = \ln(80/100)$, $r_6 = \ln(100/80)$, $r_7 = \ln(80/64)$, and $r_8 = \ln(64/80)$. Note that four of them are $\ln(1.25)$ and the other four are $-\ln(1.25)$; in particular, their mean is zero.

The (unbiased) sample variance of the non-annualized monthly returns is

$$\frac{1}{n-1}\sum_{j=1}^{n}(r_j-\bar{r})^2 = \frac{1}{7}\sum_{j=1}^{8}(r_j-\bar{r})^2 = \frac{1}{7}\sum_{j=1}^{8}(r_j)^2 = \frac{8}{7}[\ln(1.25)]^2.$$

The annual standard deviation is related to the monthly standard deviation by formula (11.5),

$$\sigma = \frac{\sigma_h}{\sqrt{h}},$$

where $h = 1/12$. Thus, the historical volatility is

$$\sqrt{12} \times \sqrt{\frac{8}{7}} \times \ln(1.25) = 82.6\%.$$

Remarks: Further discussion is given in Section 23.2 of McDonald (2006). Suppose that we observe n continuously compounded returns over the time period $[\tau, \tau + T]$. Then, $h = T/n$, and the historical annual variance of returns is estimated as

$$\frac{1}{h}\frac{1}{n-1}\sum_{j=1}^{n}(r_j-\bar{r})^2 = \frac{1}{T}\frac{n}{n-1}\sum_{j=1}^{n}(r_j-\bar{r})^2.$$

Now,

$$\bar{r} = \frac{1}{n}\sum_{j=1}^{n}r_j = \frac{1}{n}\ln\frac{S(\tau+T)}{S(\tau)},$$

which is close to zero when n is large. Thus, a simpler estimation formula is

$\dfrac{1}{h}\dfrac{1}{n-1}\sum_{j=1}^{n}(r_j)^2$ which is formula (23.2) on page 744, or equivalently, $\dfrac{1}{T}\dfrac{n}{n-1}\sum_{j=1}^{n}(r_j)^2$

which is the formula in footnote 9 on page 756. The last formula is related to #10 in this set of sample problems: With probability 1,

$$\lim_{n\to\infty}\sum_{j=1}^{n}[\ln S(jT/n) - \ln S((j-1)T/n)]^2 = \sigma^2 T.$$

18. A market-maker sells 1,000 1-year European gap call options, and delta-hedges the position with shares.

You are given:

(i) Each gap call option is written on 1 share of a nondividend-paying stock.
(ii) The current price of the stock is 100.
(iii) The stock's volatility is 100%.
(iv) Each gap call option has a strike price of 130.
(v) Each gap call option has a payment trigger of 100.
(vi) The risk-free interest rate is 0%.

Under the Black-Scholes framework, determine the initial number of shares in the delta-hedge.

(A) 586
(B) 594
(C) 684
(D) 692
(E) 797

Solution to (18)   Answer: (A)

Note that, in this problem, $r = 0$ and $\delta = 0$.

By formula (14.15) in McDonald (2006), the time-0 price of the gap option is
$$C_{gap} = SN(d_1) - 130N(d_2) = [SN(d_1) - 100N(d_2)] - 30N(d_2) = C - 30N(d_2),$$
where d_1 and d_2 are calculated with $K = 100$ (and $r = \delta = 0$) and $T = 1$, and C denotes the time-0 price of the plain-vanilla call option with exercise price 100.

In the Black-Scholes framework, delta of a derivative security of a stock is the partial derivative of the security price with respect to the stock price. Thus,
$$\Delta_{gap} = \frac{\partial}{\partial S}C_{gap} = \frac{\partial}{\partial S}C - 30\frac{\partial}{\partial S}N(d_2) = \Delta_C - 30N'(d_2)\frac{\partial}{\partial S}d_2$$
$$= N(d_1) - 30N'(d_2)\frac{1}{S\sigma\sqrt{T}},$$
where $N'(x) = \dfrac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$ is the density function of the standard normal.

Now, with $S = K = 100$, $T = 1$, and $\sigma = 1$,
$$d_1 = [\ln(S/K) + \sigma^2 T/2]/(\sigma\sqrt{T}) = (\sigma^2 T/2)/(\sigma\sqrt{T}) = \tfrac{1}{2}\sigma\sqrt{T} = \tfrac{1}{2},$$
and $d_2 = d_1 - \sigma\sqrt{T} = -\tfrac{1}{2}$. Hence, at time 0
$$\Delta_{gap} = N(d_1) - 30N'(d_2)\frac{1}{100} = N(\tfrac{1}{2}) - 0.3N'(-\tfrac{1}{2}) = N(\tfrac{1}{2}) - 0.3\frac{1}{\sqrt{2\pi}}\, e^{-(-\frac{1}{2})^2/2}$$
$$= 0.6915 - 0.3\frac{0.8825}{\sqrt{2\pi}} = 0.6915 - 0.3\times0.352 = 0.6915 - 0.1056 = 0.5859$$

Remark: The formula for the standard normal density function, $\dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$, can be found in the Normal Table distributed to students.

19. Consider a *forward start option* which, 1 year from today, will give its owner a 1-year European call option with a strike price equal to the stock price at that time.

You are given:

(i) The European call option is on a stock that pays no dividends.

(ii) The stock's volatility is 30%.

(iii) The forward price for delivery of 1 share of the stock 1 year from today is 100.

(iv) The continuously compounded risk-free interest rate is 8%.

Under the Black-Scholes framework, determine the price today of the forward start option.

(A) 11.90

(B) 13.10

(C) 14.50

(D) 15.70

(E) 16.80

Solution to (19)   Answer: (C)

This problem is based on Exercise 14.21 on page 465 of McDonald (2006).

Let S_1 denote the stock price at the end of one year. Apply the Black-Scholes formula to calculate the price of the at-the-money call one year from today, conditioning on S_1.

$d_1 = [\ln (S_1/S_1) + (r + \sigma^2/2)T]/(\sigma\sqrt{T}) = (r + \sigma^2/2)/\sigma = 0.417$, which turns out to be independent of S_1.

$d_2 = d_1 - \sigma\sqrt{T} = d_1 - \sigma = 0.117$

The value of the forward start option at time 1 is

$$C(S_1) = S_1 N(d_1) - S_1 e^{-r} N(d_2)$$
$$= S_1[N(0.417) - e^{-0.08} N(0.117)]$$
$$\approx S_1[N(0.42) - e^{-0.08} N(0.12)]$$
$$= S_1[0.6628 - e^{-0.08} \times 0.5438]$$
$$= 0.157 S_1.$$

(Note that, when viewed from time 0, S_1 is a random variable.)

Thus, the time-0 price of the forward start option must be 0.157 multiplied by the time-0 price of a security that gives S_1 as payoff at time 1, i.e., multiplied by the prepaid forward price $F^P_{0,1}(S)$. Hence, the time-0 price of the forward start option is

$$0.157 \times F^P_{0,1}(S) = 0.157 \times e^{-0.08} \times F_{0,1}(S) = 0.157 \times e^{-0.08} \times 100 = 14.5$$

Remark: A key to pricing the forward start option is that d_1 and d_2 turn out to be independent of the stock price. This is the case if the strike price of the call option will be set as a fixed percentage of the stock price at the issue date of the call option.

20. Assume the Black-Scholes framework. Consider a stock, and a European call option and a European put option on the stock. The current stock price, call price, and put price are 45.00, 4.45, and 1.90, respectively.

Investor A purchases two calls and one put. Investor B purchases two calls and writes three puts.

The current elasticity of Investor A's portfolio is 5.0. The current delta of Investor B's portfolio is 3.4.

Calculate the current put-option elasticity.

(A) −0.55

(B) −1.15

(C) −8.64

(D) −13.03

(E) −27.24

Solution to (20)   Answer: (D)

Applying the formula

$$\Delta_{\text{portfolio}} = \frac{\partial}{\partial S}\text{portfolio value}$$

to Investor B's portfolio yields

$$3.4 = 2\Delta_C - 3\Delta_P. \tag{1}$$

Applying the elasticity formula

$$\Omega_{\text{portfolio}} = \frac{\partial}{\partial \ln S}\ln[\text{portfolio value}] = \frac{S}{\text{portfolio value}} \times \frac{\partial}{\partial S}\text{portfolio value}$$

to Investor A's portfolio yields

$$5.0 = \frac{S}{2C + P}(2\Delta_C + \Delta_P) = \frac{45}{8.9 + 1.9}(2\Delta_C + \Delta_P),$$

or

$$1.2 = 2\Delta_C + \Delta_P. \tag{2}$$

Now, $(2) - (1) \quad \Rightarrow \quad -2.2 = 4\Delta_P.$

Hence, time-0 put option elasticity $= \Omega_P = \dfrac{S}{P}\Delta_P = \dfrac{45}{1.9} \times -\dfrac{2.2}{4} = -13.03$, which is (D).

Remarks:

(i)   If the stock pays no dividends, and if the European call and put options have the same expiration date and strike price, then $\Delta_C - \Delta_P = 1$. In this problem, the put and call do not have the same expiration date and strike price; so this relationship does not hold.

(ii)   If your copy of McDonald (2006) was printed before 2008, then you need to replace the last paragraph of Section 12.3 on page 395 by
http://www.kellogg.northwestern.edu/faculty/mcdonald/htm/erratum395.pdf
The n_i in the new paragraph corresponds to the ω_i on page 389.

(iii)   The statement on page 395 in McDonald (2006) that "[t]he elasticity of a portfolio is the *weighted average* of the elasticities of the portfolio components" may remind students, who are familiar with fixed income mathematics, the concept of *duration*. Formula (3.5.8) on page 101 of *Financial Economics*: *With Applications to Investments, Insurance and Pensions* (edited by H.H. Panjer and published by The Actuarial Foundation in 1998) shows that the so-called *Macaulay duration* is an elasticity.

(iv)   In the Black-Scholes framework, the hedge ratio or delta of a portfolio is the partial derivative of the portfolio price with respect to the stock price. In other continuous-time frameworks (which are not in the syllabus of Exam MFE/3F), the hedge ratio may not be given by a partial derivative; for an example, see formula (10.5.7) on page 478 of *Financial Economics*: *With Applications to Investments, Insurance and Pensions*.

21. The Cox-Ingersoll-Ross (CIR) interest-rate model has the short-rate process:
$$dr(t) = a[b - r(t)]dt + \sigma\sqrt{r(t)}\, dZ(t),$$
where $\{Z(t)\}$ is a standard Brownian motion.

For $t \le T$, let $P(r,t,T)$ be the price at time t of a zero-coupon bond that pays \$1 at time T, if the short-rate at time t is r. The price of each zero-coupon bond in the CIR model follows an Itô process:
$$\frac{dP[r(t),t,T]}{P[r(t),t,T]} = \alpha[r(t),t,T]dt - q[r(t),t,T]dZ(t), \qquad t \le T.$$

You are given $\alpha(0.05,\ 7,\ 9) = 0.06$.

Calculate $\alpha(0.04,\ 11,\ 13)$.

(A)   0.042

(B)   0.045

(C)   0.048

(D)   0.050

(E)   0.052

Solution to (21)   Answer: (C)

As pointed out on pages 782 and 783 of McDonald (2006), the condition of no riskless arbitrages implies that the Sharpe ratio does not depend on T,

$$\frac{\alpha(r,t,T)-r}{q(r,t,T)}=\phi(r,t). \tag{24.17}$$

(Also see Section 20.4.) This result may not seem applicable because we are given an α for $t = 7$ while asked to find an α for $t = 11$.

Now, equation (24.12) in McDonald (2006) is

$$q(r,t,T)=-\sigma(r)P_r(r,t,T)/P(r,t,T)=-\sigma(r)\frac{\partial}{\partial r}\ln[P(r,t,T)],$$

the substitution of which in (24.17) yields

$$\alpha(r,t,T)-r=-\phi(r,t)\sigma(r)\frac{\partial}{\partial r}\ln[P(r,t,T)].$$

In the CIR model (McDonald 2006, p. 787), $\sigma(r)=\sigma\sqrt{r}$, $\phi(r,t)=\dfrac{\overline{\phi}}{\sigma}\sqrt{r}$ with $\overline{\phi}$ being

a constant, and $\dfrac{\partial}{\partial r}\ln[P(r,t,T)]=-B(t,T)$. Thus,

$$\begin{aligned}
\alpha(r,t,T)-r &= -\phi(r,t)\sigma(r)\frac{\partial}{\partial r}\ln[P(r,t,T)]\\
&= -\frac{\overline{\phi}}{\sigma}\sqrt{r}\times\sigma\sqrt{r}\times[-B(t,T)]\\
&= \overline{\phi}\,rB(t,T),
\end{aligned}$$

or

$$\frac{\alpha(r,t,T)}{r}=1+\overline{\phi}\,B(t,T).$$

Because $B(t,T)$ depends on t and T through the difference $T-t$, we have, for $T_1 - t_1 = T_2 - t_2$,

$$\frac{\alpha(r_1,t_1,T_1)}{r_1}=\frac{\alpha(r_2,t_2,T_2)}{r_2}.$$

Hence,

$$\alpha(0.04,11,13)=\frac{0.04}{0.05}\alpha(0.05,7,9)=0.8\times0.06=0.048.$$

Remarks: (i) In earlier printings of McDonald (2006), the minus sign in (24.1) was given as a plus sign. Hence, there was no minus sign in (24.12) and $\overline{\phi}$ would be a negative constant. However, these changes would not affect the answer to this question. (ii) What McDonald calls Brownian motion is usually called *standard* Brownian motion by other authors.

22. You are given:

(i) The true stochastic process of the short-rate is given by
$$dr(t) = [0.09 - 0.5r(t)]dt + 0.3dZ(t),$$
where $\{Z(t)\}$ is a standard Brownian motion under the true probability measure.

(ii) The risk-neutral process of the short-rate is given by
$$dr(t) = [0.15 - 0.5r(t)]dt + \sigma(r(t))d\tilde{Z}(t),$$
where $\{\tilde{Z}(t)\}$ is a standard Brownian motion under the risk-neutral probability measure.

(iii) $g(r, t)$ denotes the price of an interest-rate derivative at time t, if the short-rate at that time is r. The interest-rate derivative does not pay any dividend or interest.

(iv) $g(r(t), t)$ satisfies
$$dg(r(t), t) = \mu(r(t), g(r(t), t))dt - 0.4g(r(t), t)dZ(t).$$

Determine $\mu(r, g)$.

(A) $(r - 0.09)g$

(B) $(r - 0.08)g$

(C) $(r - 0.03)g$

(D) $(r + 0.08)g$

(E) $(r + 0.09)g$

Solution to (22)   Answer: (D)

Formula (24.2) of McDonald (2006) is
$$dr(t) = a(r(t))\,dt + \sigma(r(t))\,dZ(t).$$
Because
$$dZ(t) = d\tilde{Z}(t) + \phi(r(t), t)dt$$
where $\phi(r, t)$ is the Sharpe ratio, we have
$$dr(t) = \left[a(r(t)) + \sigma(r(t))\phi(r(t), t)\right]dt + \sigma(r(t))d\tilde{Z}(t)$$
which is formula (24.19). By comparing (24.2) with the formula in (i), we obtain
$$a(r) = 0.09 - 0.5r$$
and
$$\sigma(r) = 0.3.$$
By comparing (24.19) with the formula in (ii), we see that
$$0.15 - 0.5r = a(r) + \sigma(r)\phi(r, t)$$
$$= [0.09 - 0.5r] + 0.3\phi(r, t),$$
or
$$\phi(r, t) = 0.2.$$
Now, for the model to be arbitrage free, the Sharpe ratio of the interest-rate derivative should also be given by $\phi(r, t)$. Rewriting the equation in (iv) as
$$\frac{dg(r(t), t)}{g(r(t), t)} = \frac{\mu(r(t), g(r(t), t))}{g(r(t), t)}dt - 0.4dZ(t) \qquad \text{[cf. equation (24.13)]}$$
and because there are no dividend or interest payments, we obtain
$$\frac{\dfrac{\mu(r, g(r,t))}{g(r,t)} - r}{0.4} = \phi(r, t) \qquad \text{[cf. equation (24.17)]}$$
$$= 0.2.$$
Thus,
$$\mu(r, g) = (r + 0.08)g.$$

Remark: Upon comparing the formula $d\tilde{Z}(t) = dZ(t) - \phi(r(t), t)dt$ with
$$d\tilde{Z}(t) = dZ(t) + \eta dt = dZ(t) + \frac{\alpha - r}{\sigma}dt,$$
which can be found on page 662 of McDonald (2006), you will note that the signs in front of the Sharpe ratios are different. The minus sign in front of $\phi(r(t), t))$ is due the minus sign in front of $q(r(t), t))$ in (24.1). [If your copy of McDonald (2006) has a plus sign in (24.1), then you have an earlier printing of the book.]

23. Consider a European call option on a nondividend-paying stock with exercise date T, $T > 0$. Let $S(t)$ be the price of one share of the stock at time t, $t \geq 0$. For $0 \leq t \leq T$, let $C(s, t)$ be the price of one unit of the call option at time t, if the stock price is s at that time. You are given:

(i) $\dfrac{dS(t)}{S(t)} = 0.1dt + \sigma dZ(t)$, where σ is a positive constant and $\{Z(t)\}$ is a Brownian motion.

(ii) $\dfrac{dC(S(t),t)}{C(S(t),t)} = \gamma(S(t),t)dt + \sigma_C(S(t),t)dZ(t), \qquad 0 \leq t \leq T$

(iii) $C(S(0), 0) = 6$.

(iv) At time $t = 0$, the cost of shares required to delta-hedge one unit of the call option is 9.

(v) The continuously compounded risk-free interest rate is 4%.

Determine $\gamma(S(0), 0)$.

(A)   0.10

(B)   0.12

(C)   0.13

(D)   0.15

(E)   0.16

Solution to (23)   Answer: (C)

Equation (21.22) of McDonald (2006) is

$$\alpha_{\text{option}} - r = \frac{SV_S}{V}(\alpha - r),$$

which, for this problem, translates to

$$\gamma(S(t),t) - 0.04 = \frac{S(t) \times \Delta(S(t),t)}{C(S(t),t)} \times (0.1 - 0.04).$$

Because

$$\frac{S(0) \times \Delta(S(0),0)}{C(S(0),0)} = \frac{9}{6} = 1.5,$$

we have

$$\gamma(S(0), 0) = 0.04 + 1.5 \times (0.1 - 0.04) = 0.13$$

(which is the time-0 continuously compounded expected rate of return on the option).

Remark: Equation (21.20) on page 687 of McDonald (2006) should be the same as (12.9) on page 393,

$$\sigma_{\text{option}} = |\Omega| \times \sigma,$$

and (21.21) should be changed to

$$\frac{\alpha - r}{\sigma} = \text{sign}(\Omega) \times \frac{\alpha_{\text{option}} - r}{\sigma_{\text{option}}}.$$

Note that Ω, α_{option}, and σ_{option} are functions of t.

24. Consider the stochastic differential equation:

$$dX(t) = \lambda[\alpha - X(t)]dt + \sigma\,dZ(t), \qquad t \geq 0,$$

where λ, α and σ are positive constants, and $\{Z(t)\}$ is a standard Brownian motion. The value of $X(0)$ is known.

Find a solution.

(A) $\quad X(t) = X(0)\,e^{-\lambda t} + \alpha(1 - e^{-\lambda t})$

(B) $\quad X(t) = X(0) + \int_0^t \alpha\,ds + \int_0^t \sigma\,dZ(s)$

(C) $\quad X(t) = X(0) + \int_0^t \alpha X(s)\,ds + \int_0^t \sigma X(s)\,dZ(s)$

(D) $\quad X(t) = X(0) + \alpha(e^{\lambda t} - 1) + \int_0^t \sigma e^{\lambda s}\,dZ(s)$

(E) $\quad X(t) = X(0)\,e^{-\lambda t} + \alpha(1 - e^{-\lambda t}) + \int_0^t \sigma e^{-\lambda(t-s)}\,dZ(s)$

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Solution to (24)   Answer: (E)

The given stochastic differential equation is (20.9) in McDonald (2006).

Rewrite the equation as

$$dX(t) \ + \ \lambda X(t)dt \ = \ \lambda\alpha dt + \sigma\,dZ(t).$$

If this were an ordinary differential equation, we would solve it by the *method of integrating factors*. (Students of life contingencies have seen the method of integrating factors in Exercise 4.22 on page 129 and Exercise 5.5 on page 158 of *Actuarial Mathematics*, 2nd edition.) Let us give this a try. Multiply the equation by the integrating factor $e^{\lambda t}$, we have

$$e^{\lambda t}\,dX(t) \ + \ e^{\lambda t}\lambda X(t)dt \ = \ e^{\lambda t}\lambda\alpha dt \ + \ e^{\lambda t}\sigma dZ(t). \qquad (*)$$

We hope that the left-hand side is exactly $d[e^{\lambda t}X(t)]$. To check this, consider $f(x, t) = e^{\lambda t}x$, whose relevant derivatives are $f_x(x, t) = e^{\lambda t}$, $f_{xx}(x, t) = 0$, and $f_t(x, t) = \lambda e^{\lambda t}x$. By Itô's Lemma,

$$df(X(t), t) \ = \ e^{\lambda t}\,dX(t) \ + \ 0 \ + \ \lambda e^{\lambda t}\,X(t)dt,$$

which is indeed the left-hand side of (*). Now, (*) can be written as

$$d[e^{\lambda s}X(s)] \ = \ \lambda\alpha e^{\lambda s}ds + \sigma e^{\lambda s}dZ(s).$$

Integrating both sides from $s = 0$ to $s = t$, we have

$$e^{\lambda t}X(t) - e^{\lambda 0}X(0) = \lambda\alpha\int_0^t e^{\lambda s}ds + \sigma\int_0^t e^{\lambda s}dZ(s) = \alpha(e^{\lambda t}-1) + \sigma\int_0^t e^{\lambda s}dZ(s),$$

or

$$e^{\lambda t}X(t) \ = \ X(0) \ + \ \alpha(e^{\lambda t}-1) \ + \ \sigma\int_0^t e^{\lambda s}dZ(s).$$

Multiplying both sides by $e^{-\lambda t}$ and rearranging yields

$$X(t) \ = \ X(0)e^{-\lambda t} + \alpha(1-e^{-\lambda t}) \ + \ \sigma e^{-\lambda t}\int_0^t e^{\lambda s}dZ(s)$$

$$= \ X(0)e^{-\lambda t} + \alpha(1-e^{-\lambda t}) \ + \ \sigma\int_0^t e^{-\lambda(t-s)}dZ(s),$$

which is (E).

Remarks: This question is the same as Exercise 20.9 on page 674. In the above, the solution is derived by solving the stochastic differential equation, while in Exercise 20.9, you are asked to use Itô's Lemma to verify that (E) satisfies the stochastic differential equation.

If $t = 0$, then the right-hand side of (E) is $X(0)$.

If $t > 0$, we differentiate (E). The first and second terms on the right-hand side are not random and have derivatives $-\lambda X(0)e^{-\lambda t}$ and $\alpha\lambda e^{-\lambda t}$, respectively. To differentiate the stochastic integral in (E), we write

$$\int_0^t e^{-\lambda(t-s)}dZ(s) = e^{-\lambda t}\int_0^t e^{\lambda s}dZ(s),$$

which is a product of a deterministic factor and a stochastic factor. Then,

$$d\left(e^{-\lambda t}\int_0^t e^{\lambda s}dZ(s)\right) = (de^{-\lambda t})\int_0^t e^{\lambda s}dZ(s) + e^{-\lambda t}d\int_0^t e^{\lambda s}dZ(s)$$

$$= (de^{-\lambda t})\int_0^t e^{\lambda s}dZ(s) + e^{-\lambda t}[e^{\lambda t}dZ(t)]$$

$$= -\left(\lambda e^{-\lambda t}\int_0^t e^{\lambda s}dZ(s)\right)dt + dZ(t).$$

Thus,

$$dX(t) = -\lambda X(0)e^{-\lambda t}dt + \alpha\lambda e^{-\lambda t}dt - \sigma\left(\lambda e^{-\lambda t}\int_0^t e^{\lambda s}dZ(s)\right)dt + \sigma dZ(t)$$

$$= -\lambda\left[X(0)e^{-\lambda t} + \sigma e^{-\lambda t}\int_0^t e^{\lambda s}dZ(s)\right]dt + \alpha\lambda e^{-\lambda t}dt + \sigma dZ(t)$$

$$= -\lambda[X(t) - \alpha(1 - e^{-\lambda t})]dt + \alpha\lambda e^{-\lambda t}dt + \sigma dZ(t)$$

$$= -\lambda[X(t) - \alpha]dt + \sigma dZ(t),$$

which is the same as the given stochastic differential equation.

25. Consider a chooser option (also known as an as-you-like-it option) on a nondividend-paying stock. At time 1, its holder will choose whether it becomes a European call option or a European put option, each of which will expire at time 3 with a strike price of $100.

The chooser option price is $20 at time $t = 0$.

The stock price is $95 at time $t = 0$. Let $C(T)$ denote the price of a European call option at time $t = 0$ on the stock expiring at time T, $T > 0$, with a strike price of $100.

You are given:

(i)   The risk-free interest rate is 0.

(ii) $C(1) = \$4$.

Determine $C(3)$.

(A) $ 9

(B) $11

(C) $13

(D) $15

(E) $17

Solution to (25)   Answer: (B)

Let $C(S(t), t, T)$ denote the price at time-t of a European call option on the stock, with exercise date T and exercise price $K = 100$. So,
$$C(T) = C(95, 0, T).$$
Similarly, let $P(S(t), t, T)$ denote the time-t put option price.

At the choice date $t = 1$, the value of the chooser option is
$$\text{Max}[C(S(1), 1, 3), P(S(1),1, 3)],$$
which can expressed as
$$C(S(1), 1, 3) + \text{Max}[0, P(S(1),1, 3) - C(S(1), 1, 3)]. \tag{1}$$
Because the stock pays no dividends and the interest rate is zero,
$$P(S(1),1, 3) - C(S(1), 1, 3) = K - S(1)$$
by put-call parity. Thus, the second term of (1) simplifies as
$$\text{Max}[0, K - S(1)],$$
which is the payoff of a European put option. As the time-1 value of the chooser option is
$$C(S(1), 1, 3) + \text{Max}[0, K - S(1)],$$
its time-0 price *must* be
$$C(S(0), 0, 3) + P(S(0), 0, 1),$$
which, by put-call parity, is
$$C(S(0), 0, 3) + [C(S(0), 0, 1) + K - S(0)]$$
$$= C(3) + [C(1) + 100 - 95] = C(3) + C(1) + 5.$$

Thus,
$$C(3) = 20 - (4 + 5) = 11.$$

Remark: The problem is a modification of Exercise 14.20.b.

**26.** Consider European and American options on a nondividend-paying stock.

You are given:

(i) All options have the same strike price of 100.

(ii) All options expire in six months.

(iii) The continuously compounded risk-free interest rate is 10%.

You are interested in the graph for the price of an option as a function of the current stock price. In each of the following four charts I–IV, the horizontal axis, S, represents the current stock price, and the vertical axis, π, represents the price of an option.

I.
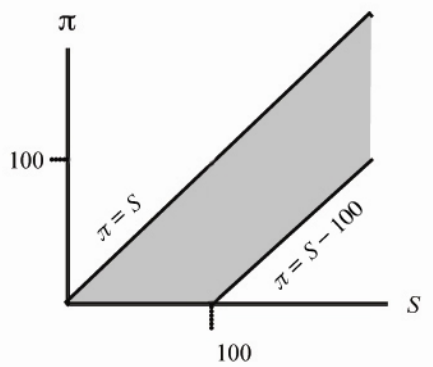
II.
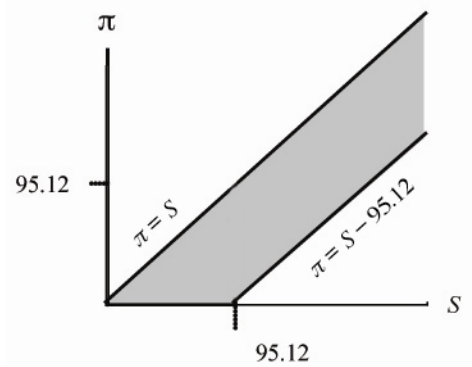
III.
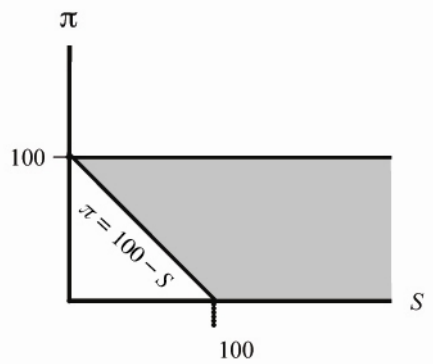
IV.
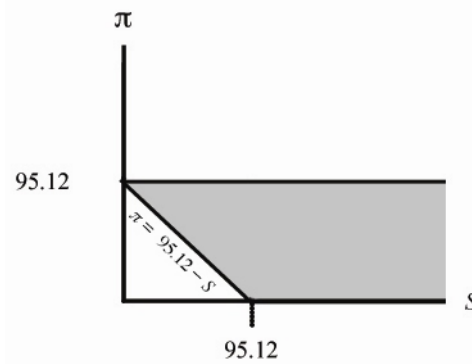
Match the option with the shaded region in which its graph lies. If there are two or more possibilities, choose the chart with the smallest shaded region.

26.   Continued

	European Call	American Call	European Put	American Put
(A)	I	I	III	III
(B)	II	I	IV	III
(C)	II	I	III	III
(D)	II	II	IV	III
(E)	II	II	IV	IV

Solution to (26)   Answer: (D)

$$T = \frac{1}{2}; \quad \mathrm{PV}_{0,T}(K) = Ke^{-rT} = 100e^{-0.1/2} = 100e^{-0.05} = 95.1229 \approx 95.12.$$

By (9.9) on page 293 of McDonald (2006), we have
$$S(0) \geq C_{Am} \geq C_{Eu} \geq \mathrm{Max}[0,\ F_{0,T}^P(S) - \mathrm{PV}_{0,T}(K)].$$

Because the stock pays no dividends, the above becomes
$$S(0) \geq C_{Am} = C_{Eu} \geq \mathrm{Max}[0,\ S(0) - \mathrm{PV}_{0,T}(K)].$$

Thus, the shaded region in II contains C_{Am} and C_{Eu}. (The shaded region in I also does, but it is a larger region.)

By (9.10) on page 294 of McDonald (2006), we have
$$K \geq P_{Am} \geq P_{Eu} \geq \mathrm{Max}[0, \mathrm{PV}_{0,T}(K) - F_{0,T}^P(S)]$$
$$= \mathrm{Max}[0, \mathrm{PV}_{0,T}(K) - S(0)]$$
because the stock pays no dividends. However, the region bounded above by $\pi = K$ and bounded below by $\pi = \mathrm{Max}[0, \mathrm{PV}_{0,T}(K) - S]$ is not given by III or IV.

Because an American option can be exercised immediately, we have a tighter lower bound for an American put,
$$P_{Am} \geq \mathrm{Max}[0, K - S(0)].$$
Thus,
$$K \geq P_{Am} \geq \mathrm{Max}[0, K - S(0)],$$
showing that the shaded region in III contains P_{Am}.

For a European put, we can use put-call parity and the inequality $S(0) \geq C_{Eu}$ to get a tighter upper bound,
$$\mathrm{PV}_{0,T}(K) \geq P_{Eu}.$$
Thus,
$$\mathrm{PV}_{0,T}(K) \geq P_{Eu} \geq \mathrm{Max}[0, \mathrm{PV}_{0,T}(K) - S(0)],$$

showing that the shaded region in IV contains P_{Eu}.

Remarks:

(i)   It turns out that II and IV can be found on page 156 of Capiński and Zastawniak (2003) *Mathematics for Finance: An Introduction to Financial Engineering*, Springer Undergraduate Mathematics Series.

(ii)   The last inequality in (9.9) can be derived as follows. By put-call parity,

$$C_{Eu} = P_{Eu} + F_{0,T}^{P}(S) - e^{-rT}K$$

$$\geq F_{0,T}^{P}(S) - e^{-rT}K \qquad \text{because } P_{Eu} \geq 0.$$

We also have

$$C_{Eu} \geq 0.$$

Thus,

$$C_{Eu} \geq \text{Max}[0,\ F_{0,T}^{P}(S) - e^{-rT}K].$$

(iii)   An alternative derivation of the inequality above is to use *Jensen's Inequality* (see, in particular, page 883).

$$C_{Eu} = E * \left[e^{-rT}\text{Max}(0, S(T) - K) \right]$$

$$\geq e^{-rT}\text{Max}(0, E * [S(T) - K]) \text{ because of Jensen's Inequality}$$

$$= \text{Max}(0, E * [e^{-rT}S(T)] - e^{-rT}K)$$

$$= \text{Max}(0, F_{0,T}^{P}(S) - e^{-rT}K).$$

Here, E* signifies risk-neutral expectation.

(iv)   That $C_{Eu} = C_{Am}$ for nondividend-paying stocks can be shown by Jensen's Inequality.

27. You are given the following information about a securities market:

(i)   There are two nondividend-paying stocks, X and Y.

(ii) The current prices for X and Y are both $100.

(iii) The continuously compounded risk-free interest rate is 10%.

(iv) There are three possible outcomes for the prices of X and Y one year from now:

Outcome	X	Y
1	$200	$0
2	$50	$0
3	$0	$300

Let C_X be the price of a European call option on X, and P_Y be the price of a European put option on Y. Both options expire in one year and have a strike price of $95.

Calculate $P_Y - C_X$.

(A) $4.30

(B) $4.45

(C) $4.59

(D) $4.75

(E) $4.94

Solution to (27)　　　　　　Answer: (A)

We are given the price information for three securities:

B:　　　　　$e^{-0.1}$　　←　　1
　　　　　　　　　　　　　　　　1
　　　　　　　　　　　　　　　　1

X:　　　　　100　　←　　200
　　　　　　　　　　　　　　50
　　　　　　　　　　　　　　0

Y:　　　　　100　　←　　0
　　　　　　　　　　　　　　0
　　　　　　　　　　　　　　300
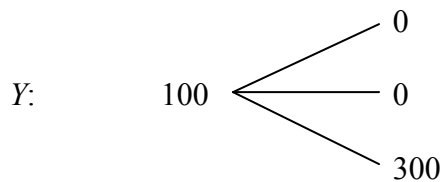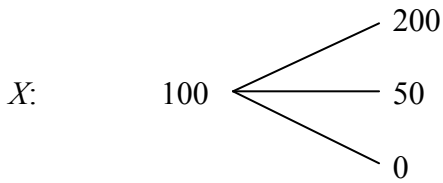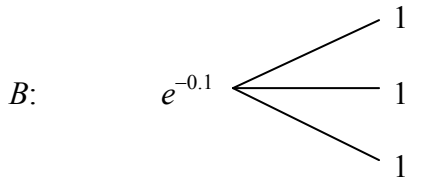
The problem is to find the price of the following security

　　　　　　　　??　　←　　−10
　　　　　　　　　　　　　95
　　　　　　　　　　　　　0
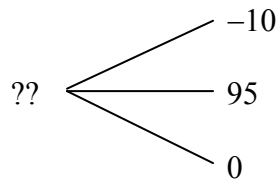
The time-1 payoffs come from:
$(95 - 0)_+ - (200 - 95)_+ = 95 - 105 = -10$
$(95 - 0)_+ - (50 - 95)_+ = 95 - 0 = 95$
$(95 - 300)_+ - (0 - 95)_+ = 0 - 0 = 0$

So, this is a *linear algebra* problem. We can take advantage of the 0's in the time-1 payoffs. By considering linear combinations of securities B and Y, we have

$B - \dfrac{Y}{300}$:　　　　　$e^{-0.1} - \frac{1}{3}$　　←　　1
　　　　　　　　　　　　　　　　　　　　　1
　　　　　　　　　　　　　　　　　　　　　0
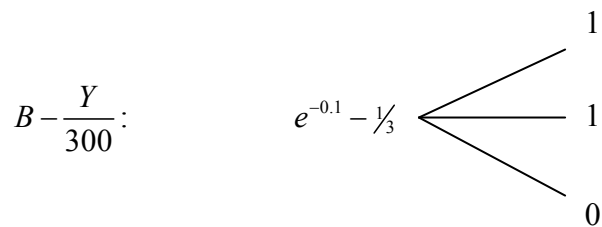
We now consider linear combinations of this security, $B - \dfrac{Y}{300}$, and X. For replicating the payoff of the put-minus-call security, the number of units of X and the number of units of $B - \dfrac{Y}{300}$ are given by

$$\begin{pmatrix} 200 & 1 \\ 50 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -10 \\ 95 \end{pmatrix}.$$

Thus, the time-0 price of the put-minus-call security is

$$(100, \quad e^{-0.1} - \tfrac{1}{3}) \begin{pmatrix} 200 & 1 \\ 50 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -10 \\ 95 \end{pmatrix}.$$

Applying the 2-by-2 matrix inversion formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

to the above, we have

$$\frac{1}{200 - 50}(100, \quad e^{-0.1} - \tfrac{1}{3}) \begin{pmatrix} 1 & -1 \\ -50 & 200 \end{pmatrix} \begin{pmatrix} -10 \\ 95 \end{pmatrix}$$

$$= \frac{1}{150}(100, \quad 0.571504085) \begin{pmatrix} -105 \\ 19500 \end{pmatrix}$$

$$= 4.295531 \approx 4.30.$$

Remarks:
(i)   We have priced the security without knowledge of the real probabilities. This is analogous to pricing options in the Black-Scholes framework without the need to know α, the continuously compounded expected return on the stock.
(ii) Matrix calculations can also be used to derive some of the results in Chapter 10 of McDonald (2006). The price of a security that pays C_u when the stock price goes up and pays C_d when the stock price goes down is

$$(S \quad 1) \begin{pmatrix} uSe^{\delta h} & e^{rh} \\ dSe^{\delta h} & e^{rh} \end{pmatrix}^{-1} \begin{pmatrix} C_u \\ C_d \end{pmatrix}$$

$$= \frac{1}{uSe^{(\delta+r)h} - dSe^{(\delta+r)h}}(S \quad 1) \begin{pmatrix} e^{rh} & -e^{rh} \\ -dSe^{\delta h} & uSe^{\delta h} \end{pmatrix} \begin{pmatrix} C_u \\ C_d \end{pmatrix}$$

$$= \frac{1}{(u-d)e^{(\delta+r)h}}(e^{rh} - de^{\delta h} \quad ue^{\delta h} - e^{rh}) \begin{pmatrix} C_u \\ C_d \end{pmatrix}$$

$$= e^{-rh}(\frac{e^{(r-\delta)h} - d}{u - d} \quad \frac{u - e^{(r-\delta)h}}{u - d}) \begin{pmatrix} C_u \\ C_d \end{pmatrix}$$

$$= e^{-rh}(p* \quad 1 - p*) \begin{pmatrix} C_u \\ C_d \end{pmatrix}.$$

(iii)   The concept of *state prices* is introduced on page 370 of McDonald (2006). A state price is the price of a security that pays 1 only when a particular state occurs. Let us denote the three states at time 1 as H, M and L, and the corresponding state prices as Q_H, Q_M and Q_L.
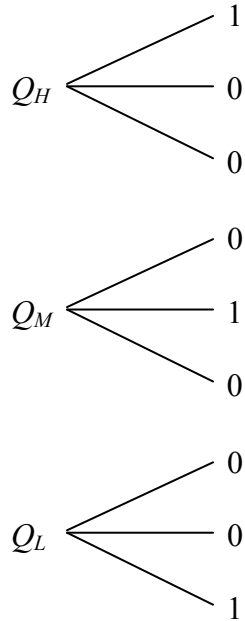
Then, the answer to the problem is
$$-10Q_H + 95Q_M + 0Q_L.$$

To find the state prices, observe that
$$\begin{cases} Q_H + Q_M + Q_L = e^{-0.1} \\ 200Q_H + 50Q_M + 0Q_L = 100 \\ 0Q_H + 0Q_M + 300Q_L = 100 \end{cases}$$

Hence,
$$(Q_H \quad Q_M \quad Q_L) = (e^{-0.1} \quad 100 \quad 100) \begin{pmatrix} 1 & 200 & 0 \\ 1 & 50 & 0 \\ 1 & 0 & 300 \end{pmatrix}^{-1} = (0.4761 \quad 0.0953 \quad 0.3333).$$

28. Assume the Black-Scholes framework. You are given:

(i) $S(t)$ is the price of a nondividend-paying stock at time t.

(ii) $S(0) = 10$

(iii) The stock's volatility is 20%.

(iv) The continuously compounded risk-free interest rate is 2%.

At time $t = 0$, you write a one-year European option that pays 100 if $[S(1)]^2$ is greater than 100 and pays nothing otherwise.

You delta-hedge your commitment.

Calculate the number of shares of the stock for your hedging program at time $t = 0$.

(A) 20

(B) 30

(C) 40

(D) 50

(E) 60

Solution to (28)   Answer: (A)

Note that $[S(1)]^2 > 100$ is equivalent to $S(1) > 10$. Thus, the option is a cash-or-nothing option with strike price 10. The time-0 price of the option is
$$100 \times e^{-rT} N(d_2).$$

To find the number of shares in the hedging program, we differentiate the price formula with respect to S,

$$\frac{\partial}{\partial S} 100 e^{-rT} N(d_2)$$

$$= 100 e^{-rT} N'(d_2) \frac{\partial d_2}{\partial S} = 100 e^{-rT} N'(d_2) \frac{1}{S\sigma\sqrt{T}}.$$

With $T = 1$, $r = 0.02$, $\delta = 0$, $\sigma = 0.2$, $S = S(0) = 10$, $K = K_2 = 10$, we have $d_2 = 0$ and

$$100 e^{-rT} N'(d_2) \frac{1}{S\sigma\sqrt{T}} = 100 e^{-0.02} N'(0) \frac{1}{2}$$

$$= 100 e^{-0.02} \frac{e^{-0^2/2}}{\sqrt{2\pi}} \frac{1}{2}$$

$$= \frac{50 e^{-0.02}}{\sqrt{2\pi}}$$

$$= 19.55.$$

29. The following is a Black-Derman-Toy binomial tree for effective annual interest rates.

Year 0	Year 1	Year 2

$$
\begin{array}{ccc}
 & & 6\% \\
 & 5\% & \\
r_0 & & r_{ud} \\
 & 3\% & \\
 & & 2\%
\end{array}
$$
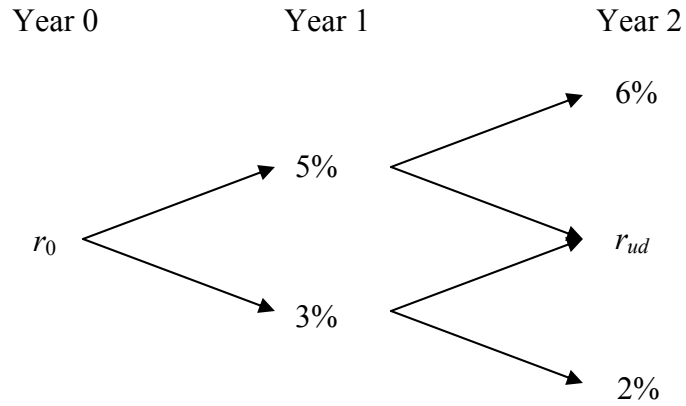
Compute the "volatility in year 1" of the 3-year zero-coupon bond generated by the tree.

(A)   14%

(B)   18%

(C)   22%

(D)   26%

(E)   30%

Solution to (29)   Answer: (D)

According to formula (24.48) on page 800 in McDonald (2006), the "volatility in year 1" of an n-year zero-coupon bond in a Black-Derman-Toy model is the number κ such that
$$y(1, n, r_u) = y(1, n, r_d)\, e^{2\kappa},$$
where y, the yield to maturity, is defined by
$$P(1, n, r) = \left(\frac{1}{1 + y(1,n,r)}\right)^{n-1}.$$
Here, $n = 3$ [and hence κ is given by the right-hand side of (24.53)]. To find $P(1, 3, r_u)$ and $P(1, 3, r_d)$, we use the method of *backward induction*.
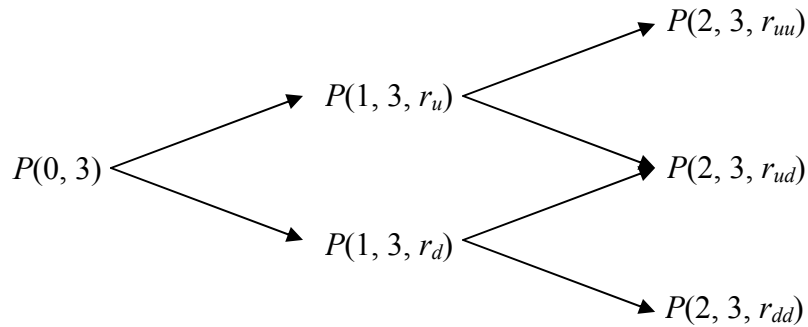
$$P(2, 3, r_{uu}) = \frac{1}{1+r_{uu}} = \frac{1}{1.06},$$

$$P(2, 3, r_{dd}) = \frac{1}{1+r_{dd}} = \frac{1}{1.02},$$

$$P(2, 3, r_{du}) = \frac{1}{1+r_{ud}} = \frac{1}{1+\sqrt{r_{uu} \times r_{dd}}} = \frac{1}{1.03464},$$

$$P(1, 3, r_u) = \frac{1}{1+r_u}\, [\tfrac{1}{2} P(2, 3, r_{uu}) + \tfrac{1}{2} P(2, 3, r_{ud})] = 0.909483,$$

$$P(1, 3, r_d) = \frac{1}{1+r_d}\, [\tfrac{1}{2} P(2, 3, r_{ud}) + \tfrac{1}{2} P(2, 3, r_{dd})] = 0.945102.$$

Hence,
$$e^{2\kappa} = \frac{y(1,3,r_u)}{y(1,3,r_d)} = \frac{[P(1,3,r_u)]^{-1/2} - 1}{[P(1,3,r_d)]^{-1/2} - 1} = \frac{0.048583}{0.028633},$$
resulting in $\kappa = 0.264348 \approx 26\%$.

Remarks: (i) The term "year n" can be ambiguous. In the Exam MLC/3L textbook *Actuarial Mathematics*, it usually means the n-th year, depicting a period of time. However, in many places in McDonald (2006), it means time n, depicting a particular instant in time. (ii) It is stated on page 799 of McDonald (2006) that "volatility in year 1" is the standard deviation of the natural log of the yield for the bond 1 year hence. This statement is vague. The concrete interpretation of "volatility in year 1" is the right-hand side of (24.48) on page 800, with $h = 1$.

30. You are given the following market data for zero-coupon bonds with a maturity payoff of $100.

Maturity (years)	Bond Price ($)	Volatility in Year 1
1	94.34	N/A
2	88.50	10%

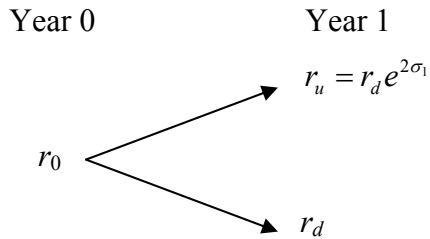A 2-period Black-Derman-Toy interest tree is calibrated using the data from above:
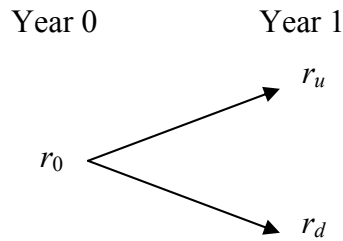
Calculate r_d, the effective annual rate in year 1 in the "down" state.

(A)   5.94%

(B)   6.60%

(C)   7.00%

(D)   7.27%

(E)   7.33%

Solution to (30)   Answer: (A)

In a BDT interest rate model, the risk-neutral probability of each "up" move is ½.

Because the "volatility in year 1" of the 2-year zero-coupon bond is 10%, we have

$$\sigma_1 = 10\%.$$

This can be seen from simplifying the right-hand side of (24.51).

We are given $P(0, 1) = 0.9434$ and $P(0, 2) = 0.8850$, and they are related as follows:

$$P(0, 2) = P(0, 1)[½P(1, 2, r_u) + ½P(1, 2, r_d)]$$

$$= P(0, 1)\left[\frac{1}{2}\frac{1}{1+r_u} + \frac{1}{2}\frac{1}{1+r_d}\right]$$

$$= P(0, 1)\left[\frac{1}{2}\frac{1}{1+r_d e^{0.2}} + \frac{1}{2}\frac{1}{1+r_d}\right].$$

Thus,

$$\frac{1}{1+r_d e^{0.2}} + \frac{1}{1+r_d} = \frac{2\times 0.8850}{0.9434} = 1.8762,$$

or

$$2 + r_d(1+e^{0.2}) = 1.8762[1 + r_d(1+e^{0.2}) + r_d^2 e^{0.2}],$$

which is equivalent to

$$1.8762e^{0.2}r_d^2 + 0.8762(1+e^{0.2})r_d - 0.1238 = 0.$$

The solution set of the quadratic equation is $\{0.0594, -0.9088\}$. Hence,

$$r_d \approx 5.94\%.$$

31. You compute the current delta for a 50-60 bull spread with the following information:

(i)   The continuously compounded risk-free rate is 5%.

(ii) The underlying stock pays no dividends.

(iii) The current stock price is $50 per share.

(iv) The stock's volatility is 20%.

(v)   The time to expiration is 3 months.

How much does delta change after 1 month, if the stock price does not change?

(A) increases by 0.04

(B) increases by 0.02

(C) does not change, within rounding to 0.01

(D) decreases by 0.02

(E) decreases by 0.04

Solution to (31)   Answer: (B)

Assume that the bull spread is constructed by buying a 50-strike call and selling a 60-strike call. (You may also assume that the spread is constructed by buying a 50-strike put and selling a 60-strike put.)

Delta for the bull spread is equal to

(delta for the 50-strike call) − (delta for the 60-strike call).

(You get the same delta value, if put options are used instead of call options.)

Call option delta = $N(d_1)$, where $d_1 = \dfrac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$

50-strike call:

$$d_1 = \frac{\ln(50/50) + (0.05 + \frac{1}{2} \times 0.2^2)(3/12)}{0.2\sqrt{3/12}} = 0.175, \quad N(d_1) \approx N(0.18) = 0.5714$$

60-strike call:

$$d_1 = \frac{\ln(50/60) + (0.05 + \frac{1}{2} \times 0.2^2)(3/12)}{0.2\sqrt{3/12}} = -1.6482, \quad N(d_1) \approx N(-1.65)$$

$$= 1 - 0.9505 = 0.0495$$

Delta of the bull spread = 0.5714 − 0.0495 = 0.5219.

After one month, 50-strike call:

$$d_1 = \frac{\ln(50/50) + (0.05 + \frac{1}{2} \times 0.2^2)(2/12)}{0.2\sqrt{2/12}} = 0.1429, \qquad N(d_1) \approx N(0.14) = 0.5557$$

60-strike call:

$$d_1 = \frac{\ln(50/60) + (0.05 + \frac{1}{2} \times 0.2^2)(2/12)}{0.2\sqrt{2/12}} = -2.0901, \qquad N(d_1) \approx N(-2.09)$$

$$= 1 - 0.9817 = 0.0183$$

Delta of the bull spread after one month = 0.5557 − 0.0183 = 0.5374.

The change in delta = 0.5374 − 0.5219 = 0.0155 ≈ 0.02.

32. At time $t = 0$, Jane invests the amount of $W(0)$ in a mutual fund. The mutual fund employs a proportional investment strategy: There is a fixed real number φ, such that, at every point of time, $100φ\%$ of the fund's assets are invested in a nondividend paying stock and $100(1 - φ)\%$ in a risk-free asset.

You are given:
(i)   The continuously compounded rate of return on the risk-free asset is r.
(ii) The price of the stock, $S(t)$, follows a geometric Brownian motion,

$$\frac{dS(t)}{S(t)} = \alpha\,dt + \sigma\,dZ(t), \qquad t \geq 0,$$

   where $\{Z(t)\}$ is a standard Brownian motion.

Let $W(t)$ denote the Jane's fund value at time t, $t \geq 0$.

Which of the following equations is true?

(A) $\dfrac{dW(t)}{W(t)} = [φ\alpha + (1 - φ)r]dt + \sigma dZ(t)$

(B) $W(t) = W(0)\exp\{[φ\alpha + (1 - φ)r]t + φ\sigma Z(t)\}$

(C) $W(t) = W(0)\exp\{[φ\alpha + (1 - φ)r - \tfrac{1}{2}φ\sigma^2]t + φ\sigma Z(t)\}$

(D) $W(t) = W(0)[S(t)/S(0)]^{φ}\,e^{(1 - φ)rt}$

(E) $W(t) = W(0)[S(t)/S(0)]^{φ}\exp[(1 - φ)(r + \tfrac{1}{2}φ\sigma^2)t]$

Solution to (32)   Answer: (E)

A proportional investment strategy means that the mutual fund's portfolio is continuously re-balanced. There is an implicit assumption that there are no transaction costs.

For $t \geq 0$, the rate of return over the time interval from t to $t + dt$ is

$$\frac{dW(t)}{W(t)} = \varphi \frac{dS(t)}{S(t)} + (1 - \varphi)rdt$$
$$= \varphi[\alpha dt + \sigma dZ(t)] + (1 - \varphi)rdt$$
$$= [\varphi\alpha + (1 - \varphi)r]dt + \varphi\sigma dZ(t). \tag{1}$$

We know
$$S(t) = S(0)\exp[(\alpha - \tfrac{1}{2}\sigma^2)t + \sigma Z(t)]. \tag{2}$$
The solution to (1) is similar,
$$W(t) = W(0)\exp\{[\varphi\alpha + (1 - \varphi)r - \tfrac{1}{2}(\varphi\sigma)^2]t + \varphi\sigma Z(t)\}. \tag{3}$$

Raising equation (2) to power φ and applying it to (3) yields
$$W(t) = W(0)[S(t)/S(0)]^\varphi \exp\{[(1 - \varphi)r - \tfrac{1}{2}(\varphi\sigma)^2 + \tfrac{1}{2}\varphi\sigma^2]t\}$$
$$= W(0)[S(t)/S(0)]^\varphi \exp[(1 - \varphi)(r + \tfrac{1}{2}\varphi\sigma^2)t], \tag{4}$$
which is (E).

Remarks:
(i) There is no restriction that the proportionality constant φ is to be between 0 and 1. If $\varphi < 0$, the mutual fund shorts the stock; if $\varphi > 1$, the mutual fund borrows money to buy more shares of the stock.

(ii) If the stock pays dividends continuously, with amount $\delta S(t)dt$ between time t and time $t+dt$, then we have equation (20.25) of McDonald (2006),
$$\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t),$$
whose solution is
$$S(t) = S(0)\exp[(\alpha - \delta - \tfrac{1}{2}\sigma^2)t + \sigma Z(t)]. \tag{5}$$
Since
$$\frac{dW(t)}{W(t)} = \varphi\left[\frac{dS(t)}{S(t)} + \delta dt\right] + (1 - \varphi)rdt$$
$$= \varphi[(\alpha - \delta)dt + \sigma dZ(t) + \delta dt] + (1 - \varphi)rdt$$
$$= [\varphi\alpha + (1 - \varphi)r]dt + \varphi\sigma dZ(t),$$
formula (3) remains valid. Raising equation (5) to power φ and applying it to (3) yields
$$W(t) = W(0)[S(t)/S(0)]^\varphi \exp\{[(1 - \varphi)r - \tfrac{1}{2}(\varphi\sigma)^2 + \varphi\delta + \tfrac{1}{2}\varphi\sigma^2)]t\}$$
$$= W(0)[S(t)/S(0)]^\varphi \exp\{[\varphi\delta + (1 - \varphi)(r + \tfrac{1}{2}\varphi\sigma^2)]t\}. \tag{6}$$

Note that as in (4), $Z(t)$ and α do not appear explicitly in (6). As a check for the validity of (6), let us verify that
$$F^P_{0,t}[W(t)] = W(0). \tag{7}$$
Since
$$F^P_{0,t}[W(t)] = W(0)S(0)^{-\varphi}\exp\{[\varphi\delta + (1 - \varphi)(r + \tfrac{1}{2}\varphi\sigma^2)]t\} F^P_{0,t}[S(t)^{\varphi}],$$
equation (7) immediately follows from (20.30) of McDonald (2006).

(iii) It follows from (6) that
$$W(t) = W(0)[S(t)/S(0)]^{\varphi}$$
if and only if φ is a solution of the quadratic equation
$$\varphi\delta + (1 - \varphi)(r + \tfrac{1}{2}\varphi\sigma^2) = 0. \tag{8}$$
The solutions of (8) are $\varphi = h_1 > 1$ and $\varphi = h_2 < 0$ as defined in Section 12.6. Section 12.6 is not currently in the syllabus of Exam MFE/3F.

(iv) Another way to write (6) is
$$W(t) = W(0)[e^{\delta t}S(t)/S(0)]^{\varphi} [e^{rt}]^{(1-\varphi)} \exp[\tfrac{1}{2}\varphi(1 - \varphi)\sigma^2 t].$$

33. You own one share of a nondividend-paying stock. Because you worry that its price may drop over the next year, you decide to employ a *rolling insurance strategy*, which entails obtaining one 3-month European put option on the stock every three months, with the first one being bought immediately.

You are given:

(i)   The continuously compounded risk-free interest rate is 8%.

(ii) The stock's volatility is 30%.

(iii) The current stock price is 45.

(iv) The strike price for each option is 90% of the then-current stock price.

Your broker will sell you the four options but will charge you for their total cost now.

Under the Black-Scholes framework, how much do you now pay your broker?

(A) 1.59

(B) 2.24

(C) 2.85

(D) 3.48

(E) 3.61

Solution to (33) Answer: (C)

The problem is a variation of Exercise 14.22, whose solution uses the concept of the forward start option in Exercise 14.21.

Let us first calculate the current price of a 3-month European put with strike price being 90% of the current stock price S.

With $K = 0.9 \times S$, $r = 0.08$, $\sigma = 0.3$, and $T = \frac{1}{4}$, we have

$$d_1 = \frac{\ln(S/0.9S) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{-\ln(0.9) + (0.08 + \frac{1}{2} \times 0.09) \times \frac{1}{4}}{0.3\sqrt{\frac{1}{4}}} = 0.9107$$

$d_2 = d_1 - \sigma\sqrt{T} = d_1 - 0.3\sqrt{\frac{1}{4}} = 0.7607$

$N(-d_1) \approx N(-0.91) = 1 - N(0.91) = 1 - 0.8186 = 0.1814$

$N(-d_2) \approx N(-0.76) = 1 - N(0.76) = 1 - 0.7764 = 0.2236$

Put price $= Ke^{-rT}N(-d_2) - SN(-d_1) = 0.9Se^{-0.08 \times 0.25} \times 0.2236 - S \times 0.1814 = 0.0159S$

For the rolling insurance strategy, four put options are needed. Their costs are $0.0159S(0)$ at time 0, $0.0159S(\frac{1}{4})$ at time $\frac{1}{4}$, $0.0159S(\frac{1}{2})$ at time $\frac{1}{2}$, and $0.0159S(\frac{3}{4})$ at time $\frac{3}{4}$. Their total price at time 0 is the sum of their prepaid forward prices.

Since the stock pays no dividends, we have
$$F_{0,T}^P(S(T)) = S(0), \qquad \text{for all } T \geq 0.$$
Hence, the sum of the four prepaid forward prices is
$$0.0159S(0) \times 4 = 0.0159 \times 45 \times 4 = 2.85.$$

34. The cubic variation of the standard Brownian motion for the time interval $[0, T]$ is defined analogously to the quadratic variation as

$$\lim_{n\to\infty} \sum_{j=1}^{n} \{Z[jh]-Z[(j-1)h]\}^3 ,$$

where $h = T/n$.

What is the distribution of the cubic variation?

(A) $\mathcal{N}(0, 0)$

(B) $\mathcal{N}(0, T^{1/2})$

(C) $\mathcal{N}(0, T)$

(D) $\mathcal{N}(0, T^{3/2})$

(E) $\mathcal{N}(-\sqrt{T/2}, T)$

Solution to (34) Answer: (A)

It is stated on page 653 of McDonald (2006) that "higher-order [than quadratic] variations are zero."

Let us change the last formula on page 652 by using an exponent of 3:

$$\lim_{n\to\infty}\sum_{j=1}^{n}\{Z[jh]-Z[(j-1)h]\}^3$$

$$=\lim_{n\to\infty}\sum_{j=1}^{n}(\sqrt{h}Y_{jh})^3$$

$$=\lim_{n\to\infty}\sum_{j=1}^{n}h^{3/2}Y_{jh}^3$$

$$=\lim_{n\to\infty}\sum_{j=1}^{n}(T/n)^{3/2}(\pm1)^3.$$

Taking absolute value, we have

$$\left|\sum_{j=1}^{n}(T/n)^{3/2}(\pm1)^3\right|\le\sum_{j=1}^{n}\left|(T/n)^{3/2}(\pm1)^3\right|=\sum_{j=1}^{n}(T/n)^{3/2}=\frac{T^{3/2}}{n^{1/2}}.$$

Thus,

$$\lim_{n\to\infty}\sum_{j=1}^{n}\{Z[jh]-Z[(j-1)h]\}^3=0.$$

Alternative argument:

$$\lim_{n\to\infty}\sum_{j=1}^{n}\{Z[jh]-Z[(j-1)h]\}^3=\int_0^T[dZ(t)]^3.$$

Now,
$$[dZ(t)]^3=[dZ(t)]^2\times dZ(t)$$
$$=dt\times dZ(t) \qquad\qquad \because (20.17c)$$
$$=0 \qquad\qquad \because (20.17a)$$

Hence,
$$\int_0^T[dZ(t)]^3=\int_0^T 0=0.$$

35. The stochastic process $\{R(t)\}$ is given by

$$R(t) = R(0)e^{-t} + 0.05(1-e^{-t}) + 0.1\int_0^t e^{s-t}\sqrt{R(s)}dZ(s),$$

where $\{Z(t)\}$ is a standard Brownian motion.

Define $X(t) = [R(t)]^2$.

Find $dX(t)$.

(A) $\left[0.1\sqrt{X(t)} - 2X(t)\right]dt + 0.2[X(t)]^{3/4}dZ(t)$

(B) $\left[0.11\sqrt{X(t)} - 2X(t)\right]dt + 0.2[X(t)]^{3/4}dZ(t)$

(C) $\left[0.12\sqrt{X(t)} - 2X(t)\right]dt + 0.2[X(t)]^{3/4}dX(t)$

(D) $\left\{0.01 + [0.1 - 2R(0)]e^{-t}\right\}\sqrt{X(t)}dt + 0.2[X(t)]^{3/4}dZ(t)$

(E) $[0.1 - 2R(0)]e^{-t}\sqrt{X(t)}dt + 0.2[X(t)]^{3/4}dZ(t)$

Solution to (35)   Answer: (B)

By Itô's lemma, $dX(t) = 2R(t)dR(t) + [dR(t)]^2$.

To find $dR(t)$, write the integral $\int_0^t e^{s-t}\sqrt{R(s)}dZ(s)$ as $e^{-t}\int_0^t e^s \sqrt{R(s)}dZ(s)$.

Then,

$$dR(t) = -R(0)e^{-t}dt + 0.05e^{-t}dt - 0.1e^{-t}dt\int_0^t e^s\sqrt{R(s)}dZ(s) + 0.1e^{-t}e^t\sqrt{R(t)}dZ(t)$$

$$= -R(0)e^{-t}dt + 0.05e^{-t}dt - [R(t) - R(0)e^{-t} - 0.05(1-e^{-t})]dt + 0.1\sqrt{R(t)}dZ(t)$$

$$= [0.05 - R(t)]dt + 0.1\sqrt{R(t)}dZ(t).$$

(The above shows that $R(t)$ can be interpreted as a C-I-R short-rate.)

Thus,

$$[dR(t)]^2 = [0.1\sqrt{R(t)}]^2\,dt = 0.01R(t)dt,$$

and

$$dX(t) = 2R(t)\{[0.05 - R(t)]dt + 0.1\sqrt{R(t)}dZ(t)\} + 0.01R(t)dt$$

$$= \{0.11R(t) - 2[R(t)]^2\}dt + 0.2[R(t)]^{3/2}dZ(t)$$

$$= \left[0.11\sqrt{X(t)} - 2X(t)\right]dt + 0.2[X(t)]^{3/4}dZ(t).$$

$\Rightarrow$ Answer is (B).

Remark: This question is a version of Exercise 20.9 (McDonald 2006, p. 675).

36. Assume the Black-Scholes framework. Consider a derivative security of a stock.
You are given:

(i) The continuously compounded risk-free interest rate is 0.04.

(ii) The volatility of the stock is σ.

(iii) The stock does not pay dividends.

(iv) The derivative security also does not pay dividends.

(v) $S(t)$ denotes the time-t price of the stock.

(iv) The time-t price of the derivative security is $[S(t)]^{-k/\sigma^2}$, where k is a positive constant.

Find k.

(A)   0.04
(B)   0.05
(C)   0.06
(D)   0.07
(E)   0.08

Solution to (36)   Answer: (E)

We are given that the time-t price of the derivative security is of the form
$$V[S(t), t] = [S(t)]^a,$$
where a is a negative constant.

The function V must satisfy the Black-Scholes partial differential equation (21.11)
$$\frac{\partial V}{\partial t} + (r - \delta)S \frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial s^2} = rV .$$
Here, $\delta = 0$ because the stock does not pay dividends.

Because $V(s, t) = s^a$, we have $V_t = 0$, $V_s = as^{a-1}$, $V_{ss} = a(a-1)s^{a-2}$. Thus,
$$0 + (r - 0)S\left(aS^{a-1}\right) + \frac{1}{2}\sigma^2 S^2 \left(a(a-1)S^{a-2}\right) = rS^a,$$

or
$$ra + \frac{1}{2}\sigma^2 a(a-1) = r,$$

which is a quadratic equation of a. One obvious solution is $a = 1$ (which is not negative). The other solutions is
$$a = -\frac{2r}{\sigma^2}.$$
Consequently, $k = 2r = 2(0.04) = 0.08$.

Alternative Solution:
Let $V[S(t), t]$ denote the time-t price of a derivative security that does not pay dividends. Then, for $t \le T$,
$$V[S(t), t] = F_{t,T}^P(V[S(T), T]) .$$

In particular,
$$V[S(0), 0] = F_{0,T}^P(V[S(T), T]) .$$

We are given that $V[S(t), t] = [S(t)]^a$, where $a = -k/\sigma^2$. Thus, the equation above is
$$[S(0)]^a = F_{0,T}^P([S(T)]^a)$$
$$= e^{-rT}[S(0)]^a \exp\{[a(r - \delta) + \tfrac{1}{2}a(a - 1)\sigma^2]T\}$$
by (20.30). Hence we have the following quadratic equation for a:
$$-r + a(r - \delta) + \tfrac{1}{2}a(a - 1)\sigma^2 = 0.$$
whose solutions, with $\delta = 0$, are $a = 1$ and $a = -2r/\sigma^2$.

Remarks:

(i)   If $\delta > 0$, the solutions of the quadratic equation are $a = h_1 > 1$ and $a = h_2 < 0$ as defined in Section 12.6 of McDonald (2006). Section 12.6 is not currently in the syllabus of Exam MFE/3F.

(ii)   For those who know *martingale* theory, the alternative solution above is equivalent to seeking a such that, under the risk-neutral probability measure, the stochastic process $\{e^{-rt}[S(t)]^a; t \geq 0\}$ is a martingale.

(iii) If the derivative security pays dividends, then its price, V, does not satisfy the partial differential equation (21.11). If the dividend payment between time t and time $t + dt$ is $\Gamma(t)dt$, then the Black-Scholes equation (21.31) on page 691 will need to be modified as

$$E_t^*[dV + \Gamma(t)dt] = V \times (rdt).$$

See also Exercise 21.10 on page 700.

37. The price of a stock is governed by the stochastic differential equation:

$$\frac{dS(t)}{S(t)} = 0.03dt + 0.2dZ(t),$$

where $\{Z(t)\}$ is a standard Brownian motion. Consider the geometric average

$$G = [S(1) \times S(2) \times S(3)]^{1/3}.$$

Find the variance of $\ln[G]$.

(A) 0.03

(B) 0.04

(C) 0.05

(D) 0.06

(E) 0.07

Solution to (37)   Answer: (D)

We are to find the variance of

$$\ln G = \frac{1}{3}[\ln S(1) + \ln S(2) + \ln S(3)].$$

If

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dZ(t), \qquad t \geq 0,$$

then it follows from equation (20.29) (with $\delta = 0$) that

$$\ln S(t) = \ln S(0) + (\mu - \tfrac{1}{2}\sigma^2)t + \sigma Z(t), \qquad t \geq 0.$$

Hence,

$$\mathrm{Var}[\ln G] = \frac{1}{3^2} \mathrm{Var}[\ln S(1) + \ln S(2) + \ln S(3)]$$

$$= \frac{\sigma^2}{9} \mathrm{Var}[Z(1) + Z(2) + Z(3)].$$

Although $Z(1)$, $Z(2)$, and $Z(3)$ are not uncorrelated random variables, the increments, $Z(1) - Z(0)$, $Z(2) - Z(1)$, and $Z(3) - Z(2)$, are independent $\mathcal{N}(0, 1)$ random variables (McDonald 2006, page 650). Put

$$Z_1 = Z(1) - Z(0) = Z(1) \qquad \text{because } Z(0) = 0,$$
$$Z_2 = Z(2) - Z(1),$$

and

$$Z_3 = Z(3) - Z(2).$$

Then,

$$Z(1) + Z(2) + Z(3) = 3Z_1 + 2Z_2 + 1Z_3.$$

Thus,

$$\mathrm{Var}[\ln G] = \frac{\sigma^2}{9}[\mathrm{Var}(3Z_1) + \mathrm{Var}(2Z_2) + \mathrm{Var}(Z_3)]$$

$$= \frac{\sigma^2}{9}[3^2 + 2^2 + 1^2] = \frac{14\sigma^2}{9} = \frac{14 \times (0.2)^2}{9} = 0.06222 \approx 0.06.$$

Remarks:
(i) Consider the more general geometric average which uses N equally spaced stock prices from 0 to T, with the first price observation at time T/N,

$$G = \left[\prod_{j=1}^{N} S(jT/N)\right]^{1/N}.$$

Then,

$$\mathrm{Var}[\ln G] = \mathrm{Var}\left[\frac{1}{N}\sum_{j=1}^{N}\ln S(jT/N)\right] = \frac{\sigma^2}{N^2}\mathrm{Var}\left[\sum_{j=1}^{N} Z(jT/N)\right].$$

With the definition

$$Z_j = Z(jT/N) - Z((j-1)T/N), \; j = 1, 2, \dots, N,$$

we have

$$\sum_{j=1}^{N} Z(jT/N) = \sum_{j=1}^{N} (N+1-j)Z_j.$$

Because $\{Z_j\}$ are independent $\mathcal{N}(0, T/N)$ random variables, we obtain

$$\begin{aligned}
\operatorname{Var}[\ln G] &= \frac{\sigma^2}{N^2} \sum_{j=1}^{N} (N+1-j)^2 \operatorname{Var}[Z_j] \\
&= \frac{\sigma^2}{N^2} \sum_{j=1}^{N} (N+1-j)^2 \times \frac{T}{N} \\
&= \frac{\sigma^2}{N^2} \frac{N(N+1)(2N+1)}{6} \frac{T}{N} \\
&= \frac{(N+1)(2N+1)\sigma^2 T}{6N^2},
\end{aligned}$$

which can be checked using formula (14.19) on page 466.

(ii) Since $\ln G = \dfrac{1}{N} \sum_{j=1}^{N} \ln S(jT/N)$ is a normal random variable, the random variable G is

a lognormal random variable. The mean of $\ln G$ can be similarly derived. In fact, McDonald (2006, page 466) wrote: "Deriving these results is easier than you might guess."

(iii) As N tends to infinity, G becomes

$$\exp\left[\frac{1}{T} \int_0^T \ln S(\tau)\, d\tau \right].$$

The integral of a Brownian motion, called an *integrated Brownian motion*, is treated in textbooks on stochastic processes.

(iv) The determination of the distribution of an *arithmetic* average (the above is about the distribution of a geometric average) is a very difficult problem. See footnote 3 on page 446 of McDonald (2006) and also #56 in this set of sample questions.

38. For $t \le T$, let $P(t, T, r)$ be the price at time t of a zero-coupon bond that pays \$1 at time T, if the short-rate at time t is r.

You are given:

(i) $P(t, T, r) = A(t, T) \times \exp[-B(t, T)r]$ for some functions $A(t, T)$ and $B(t, T)$.

(ii) $B(0, 3) = 2$.

Based on $P(0, 3, 0.05)$, you use the delta-gamma approximation to estimate $P(0, 3, 0.03)$, and denote the value as $P_{Est}(0, 3, 0.03)$

Find $\dfrac{P_{Est}(0,3,0.03)}{P(0,3,0.05)}$.

(A) 1.0240

(B) 1.0408

(C) 1.0416

(D) 1.0480

(E) 1.0560

Solution to (38)   Answer: (B)

The term "delta-gamma approximations for bonds" can be found on page 784 of McDonald (2006).

By Taylor series,

$$P(t, T, r_0 + \varepsilon) \approx P(t, T, r_0) + \frac{1}{1!} P_r(t, T, r_0)\varepsilon + \frac{1}{2!} P_{rr}(t, T, r_0)\varepsilon^2 + \dots,$$

where

$$P_r(t, T, r) = -A(t, T)B(t, T)e^{-B(t, T)r} = -B(t, T)P(t, T, r)$$

and

$$P_{rr}(t, T, r) = -B(t, T)P_r(t, T, r) = [B(t, T)]^2 P(t, T, r).$$

Thus,

$$\frac{P(t,T,r_0 + \varepsilon)}{P(t,T,r_0)} \approx 1 - B(t, T)\varepsilon + \tfrac{1}{2}[B(t, T)\varepsilon]^2 + \dots$$

and

$$\frac{P_{Est}(0,3,0.03)}{P(0,3,0.05)} = 1 - (2 \times -0.02) + \tfrac{1}{2}(2 \times -0.02)^2$$

$$= 1.0408$$

39. A discrete-time model is used to model both the price of a nondividend-paying stock and the short-term (risk-free) interest rate. Each period is one year.

At time 0, the stock price is $S_0 = 100$ and the effective annual interest rate is $r_0 = 5\%$.

At time 1, there are only two states of the world, denoted by u and d. The stock prices are $S_u = 110$ and $S_d = 95$. The effective annual interest rates are $r_u = 6\%$ and $r_d = 4\%$.

Let $C(K)$ be the price of a 2-year K-strike European call option on the stock.
Let $P(K)$ be the price of a 2-year K-strike European put option on the stock.

Determine $P(108) - C(108)$.

(A)   −2.85

(B)   −2.34

(C)   −2.11

(D)   −1.95

(E)   −1.08

Solution to (39)   Answer: (B)

We are given that the securities model is a discrete-time model, with each period being one year. Even though there are only two states of the world at time 1, we cannot assume that the model is binomial after time 1. However, the difference, $P(K) - C(K)$, suggests put-call parity.

From the identity
$$x_+ - (-x)_+ = x,$$
we have
$$[K - S(T)]_+ - [S(T) - K]_+ = K - S(T),$$
which yields
$$P(K) - C(K) = F^P_{0,2}(K) - F^P_{0,2}(S)$$
$$= PV_{0,2}(K) - S(0)$$
$$= K \times P(0, 2) - S(0).$$

Thus, the problem is to find $P(0, 2)$, the price of the 2-year zero-coupon bond:
$$P(0, 2) = \frac{1}{1+r_0}\left[p^* \times P(1,2,u) + (1-p^*) \times P(1,2,d)\right]$$
$$= \frac{1}{1+r_0}\left[\frac{p^*}{1+r_u} + \frac{1-p^*}{1+r_d}\right].$$

To find the risk-neutral probability p^*, we use
$$S_0 = \frac{1}{1+r_0}\left[p^* \times S_u + (1-p^*) \times S_d\right]$$
or
$$100 = \frac{1}{1.05}\left[p^* \times 110 + (1-p^*) \times 95\right].$$
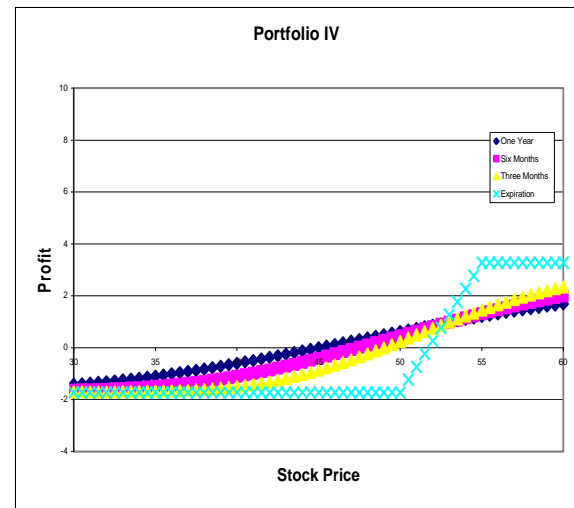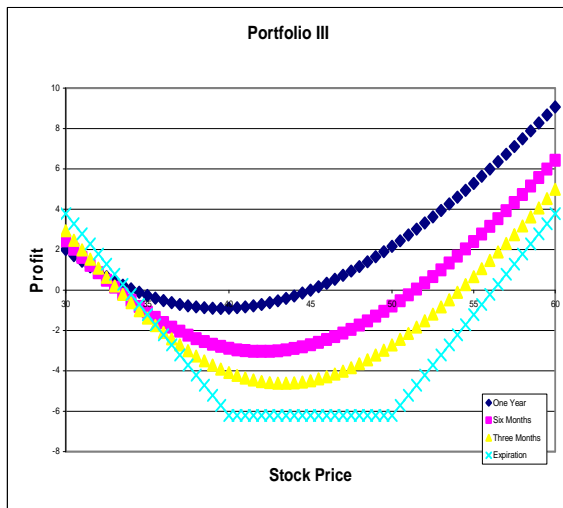This yields $p^* = \dfrac{105-95}{110-95} = \dfrac{2}{3}$, with which we obtain
$$P(0, 2) = \frac{1}{1.05}\left[\frac{2/3}{1.06} + \frac{1/3}{1.04}\right] = 0.904232.$$
Hence,
$$P(108) - C(108) = 108 \times 0.904232 - 100 = -2.34294.$$

40. The following four charts are profit diagrams for four option strategies: Bull Spread, Collar, Straddle, and Strangle. Each strategy is constructed with the purchase or sale of two 1-year European options.
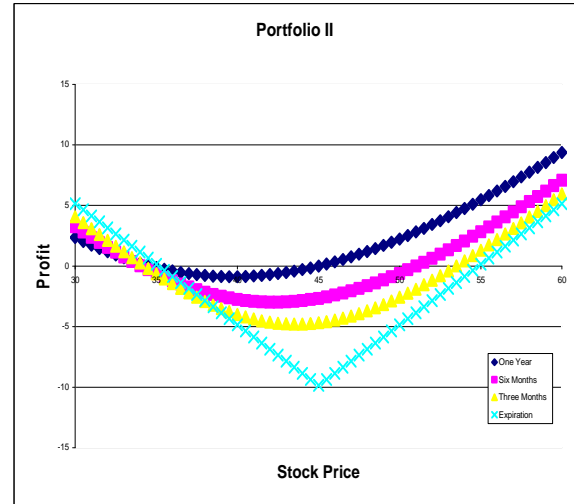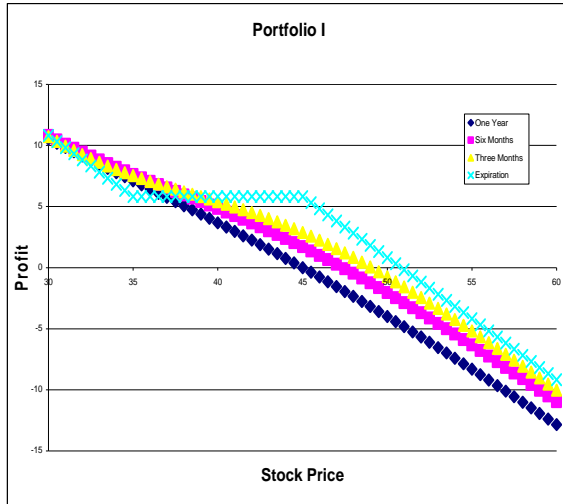
Match the charts with the option strategies.

	Bull Spread	Straddle	Strangle	Collar
(A)	I	II	III	IV
(B)	I	III	II	IV
(C)	III	IV	I	II
(D)	IV	II	III	I
(E)	IV	III	II	I

Solution to (40)   Answer: (D)

Profit diagrams are discussed Section 12.4 of McDonald (2006). Definitions of the option strategies can be found in the Glossary near the end of the textbook. See also Figure 3.17 on page 87.

The payoff function of a *straddle* is
$$\pi(s) = (K - s)_+ + (s - K)_+ = |s - K| .$$

The payoff function of a *strangle* is
$$\pi(s) = (K_1 - s)_+ + (s - K_2)_+$$
where $K_1 < K_2$.

The payoff function of a *collar* is
$$\pi(s) = (K_1 - s)_+ - (s - K_2)_+$$
where $K_1 < K_2$.

The payoff function of a *bull spread* is
$$\pi(s) = (s - K_1)_+ - (s - K_2)_+$$
where $K_1 < K_2$. Because $x_+ = (-x)_+ + x$, we have
$$\pi(s) = (K_1 - s)_+ - (K_2 - s)_+ + K_2 - K_1 .$$

The payoff function of a *bear spread* is
$$\pi(s) = (s - K_2)_+ - (s - K_1)_+$$
where $K_1 < K_2$.

41. Assume the Black-Scholes framework. Consider a 1-year European contingent claim on a stock.

You are given:

(i) The time-0 stock price is 45.

(ii) The stock's volatility is 25%.

(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.

(iv) The continuously compounded risk-free interest rate is 7%.

(v) The time-1 payoff of the contingent claim is as follows:
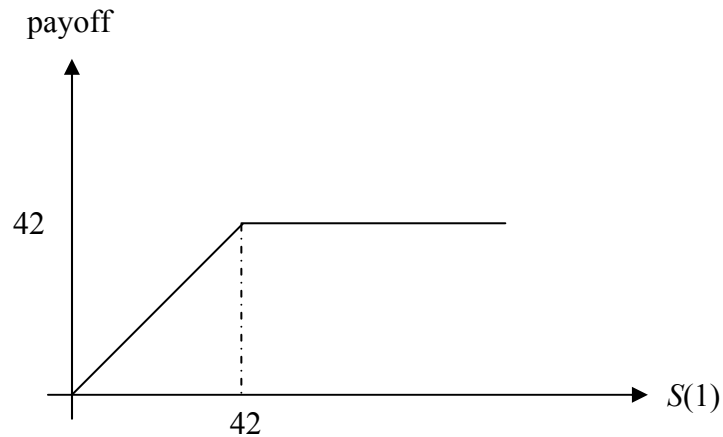
Calculate the time-0 contingent-claim elasticity.

(A) 0.24

(B) 0.29

(C) 0.34

(D) 0.39

(E) 0.44

Solution to (41)   Answer: (C)

The payoff function of the contingent claim is
$$\pi(s) \;=\; \min(42, s) \;=\; 42 + \min(0, s-42) = 42 - \max(0, 42-s) \;=\; 42 - (42-s)_+$$

The time-0 price of the contingent claim is
$$V(0) \;=\; F_{0,1}^P[\pi(S(1))]$$
$$= \; PV(42) \;-\; F_{0,1}^P[(42-S(1))_+]$$
$$= \; 42e^{-0.07} \;-\; P(45, 42, 0.25, 0.07, 1, 0.03).$$

We have $d_1 \approx 0.56$ and $d_2 \approx 0.31$, giving $N(-d_1) \approx 0.2877$ and $N(-d_2) \approx 0.3783$.

Hence, the time-0 put price is
$$P(45, 42, 0.25, 0.07, 1, 0.03) = 42e^{-0.07}(0.3783) - 45e^{-0.03}(0.2877) = 2.2506,$$
which implies
$$V(0) = 42e^{-0.07} - 2.2506 = 36.9099.$$

$$\text{Elasticity} \;=\; \frac{\partial \ln V}{\partial \ln S}$$
$$= \; \frac{\partial V}{\partial S} \times \frac{S}{V}$$
$$= \; \Delta_V \times \frac{S}{V}$$
$$= \; -\Delta_{\text{Put}} \times \frac{S}{V}.$$

$$\text{Time-0 elasticity} \;=\; e^{-\delta T} N(-d_1) \times \frac{S(0)}{V(0)}$$
$$= \; e^{-0.03} \times 0.2877 \times \frac{45}{36.9099}$$
$$= \; 0.34.$$

Remark: We can also work with $\pi(s) = s - (s-42)_+$; then
$$V(0) = 45e^{-0.03} - C(45, 42, 0.25, 0.07, 1, 0.03)$$
and
$$\frac{\partial V}{\partial S} = e^{-\delta T} - \Delta_{\text{call}} = e^{-\delta T} - e^{-\delta T} N(d_1) = e^{-\delta T} N(-d_1).$$

42. Prices for 6-month 60-strike European up-and-out call options on a stock S are available. Below is a table of option prices with respect to various H, the level of the barrier. Here, $S(0) = 50$.

H	Price of up-and-out call
60	0
70	0.1294
80	0.7583
90	1.6616
∞	4.0861

Consider a special 6-month 60-strike European "knock-in, partial knock-out" call option that knocks in at $H_1 = 70$, and "partially" knocks out at $H_2 = 80$. The strike price of the option is 60. The following table summarizes the payoff at the exercise date:

H_1 Not Hit	H_1 Hit	
	H_2 Not Hit	H_2 Hit
0	$2 \times \max[S(0.5) - 60,\, 0]$	$\max[S(0.5) - 60,\, 0]$

Calculate the price of the option.

(A) 0.6289

(B) 1.3872

(C) 2.1455

(D) 4.5856

(E) It cannot be determined from the information given above.

Solution to (42)   Answer: (D)

The "knock-in, knock-out" call can be thought of as a portfolio of
   – buying 2 ordinary up-and-in call with strike 60 and barrier H_1,
   – writing 1 ordinary up-and-in call with strike 60 and barrier H_2.

Recall also that "up-and-in" call + "up-and-out" call = ordinary call.

Let the price of the ordinary call with strike 60 be p (actually it is 4.0861),
then the price of the UIC ($H_1 = 70$) is $p - 0.1294$
and the price of the UIC ($H_1 = 80$) is $p - 0.7583$.

The price of the "knock-in, knock out" call is $2(p - 0.1294) - (p - 0.7583) = 4.5856$.

Alternative Solution:
Let $M(T) = \max_{0 \le t \le T} S(t)$ be the *running maximum* of the stock price up to time T.
Let $I[.]$ denote the *indicator function*.

For various H, the first table gives the time-0 price of payoff of the form
$$I[H > M(\tfrac{1}{2})] \times [S(\tfrac{1}{2}) - 60]_+ .$$

The payoff described by the second table is

$$I[70 \le M(\tfrac{1}{2})]\{2I[80 > M(\tfrac{1}{2})] + I[80 \le M(\tfrac{1}{2})]\}[S(\tfrac{1}{2}) - 60]_+$$
$$= \{1 - I[70 > M(\tfrac{1}{2})]\}\{1 + I[80 > M(\tfrac{1}{2})]\}[S(\tfrac{1}{2}) - 60]_+$$
$$= \{1 - I[70 > M(\tfrac{1}{2})] + I[80 > M(\tfrac{1}{2})] - I[70 > M(\tfrac{1}{2})]I[80 > M(\tfrac{1}{2})]\}[S(\tfrac{1}{2}) - 60]_+$$
$$= \{1 - 2I[70 > M(\tfrac{1}{2})] + I[80 > M(\tfrac{1}{2})]\}[S(\tfrac{1}{2}) - 60]_+$$
$$= \{I[\infty > M(\tfrac{1}{2})] - 2I[70 > M(\tfrac{1}{2})] + I[80 > M(\tfrac{1}{2})]\}[S(\tfrac{1}{2}) - 60]_+$$

Thus, the time-0 price of this payoff is $4.0861 - 2 \times 0.1294 + 0.7583 = 4.5856$.

43. Let $x(t)$ be the dollar/euro exchange rate at time t. That is, at time t, €1 = $\$x(t)$.

Let the constant r be the dollar-denominated continuously compounded risk-free interest rate. Let the constant $r_€$ be the euro-denominated continuously compounded risk-free interest rate.

You are given

$$\frac{dx(t)}{x(t)} = (r - r_€)dt + \sigma\, dZ(t),$$

where $\{Z(t)\}$ is a standard Brownian motion and σ is a constant.

Let $y(t)$ be the euro/dollar exchange rate at time t. Thus, $y(t) = 1/x(t)$.

Which of the following equation is true?

(A) $\quad\dfrac{dy(t)}{y(t)} = (r_€ - r)dt - \sigma\, dZ(t)$

(B) $\quad\dfrac{dy(t)}{y(t)} = (r_€ - r)dt + \sigma\, dZ(t)$

(C) $\quad\dfrac{dy(t)}{y(t)} = (r_€ - r - \tfrac{1}{2}\sigma^2)dt - \sigma\, dZ(t)$

(D) $\quad\dfrac{dy(t)}{y(t)} = (r_€ - r + \tfrac{1}{2}\sigma^2)dt + \sigma\, dZ(t)$

(E) $\quad\dfrac{dy(t)}{y(t)} = (r_€ - r + \sigma^2)dt - \sigma\, dZ(t)$

Solution to (43)　　　　Answer: (E)

Consider the function $f(x, t) = 1/x$. Then, $f_t = 0$, $f_x = -x^{-2}$, $f_{xx} = 2x^{-3}$.

By Itô's Lemma,
$$
\begin{aligned}
dy(t) &= df(x(t), t) \\
&= f_t dt + f_x dx(t) + \tfrac{1}{2} f_{xx}[dx(t)]^2 \\
&= 0 + [-x(t)^{-2}]dx(t) + \tfrac{1}{2}[2x(t)^{-3}][dx(t)]^2 \\
&= -x(t)^{-1}[dx(t)/x(t)] + x(t)^{-1}[dx(t)/x(t)]^2 \\
&= -y(t)[(r - r_{\text{€}})dt + \sigma\, dZ(t)] + y(t)[(r - r_{\text{€}})dt + \sigma\, dZ(t)]^2 \\
&= -y(t)[(r - r_{\text{€}})dt + \sigma\, dZ(t)] + y(t)[\sigma^2 dt],
\end{aligned}
$$
rearrangement of which yields
$$
\frac{dy(t)}{y(t)} = (r_{\text{€}} - r + \sigma^2)dt - \sigma\, dZ(t),
$$
which is (E).

Alternative Solution　Here, we use the correspondence between
$$
\frac{dW(t)}{W(t)} = \mu dt + v dZ(t)
$$
and
$$
W(t) = W(0)\exp[(\mu - \tfrac{1}{2}v^2)t + vZ(t)].
$$
Thus, the condition given is
$$
x(t) = x(0)\exp[(r - r_{\text{€}} - \tfrac{1}{2}\sigma^2)t + \sigma Z(t)].
$$

Because $y(t) = 1/x(t)$, we have
$$
\begin{aligned}
y(t) &= y(0)\exp\{-[(r - r_{\text{€}} - \tfrac{1}{2}\sigma^2)t + \sigma Z(t)]\} \\
&= y(0)\exp[(r_{\text{€}} - r + \sigma^2 - \tfrac{1}{2}(-\sigma)^2)t + (-\sigma)Z(t)],
\end{aligned}
$$
which is (E).

Remarks:
The equation

$$
\frac{dx(t)}{x(t)} = (r - r_{\text{€}})dt + \sigma\, dZ(t)
$$

can be understood in the following way. Suppose that, at time t, an investor pays \$$x(t)$ to purchase €1. Then, his instantaneous profit is the sum of two quantities:

(1) instantaneous change in the exchange rate, $[x(t+dt) - x(t)]$, or \$ $dx(t)$,

(2) € $r_{\text{€}}dt$, which is the instantaneous interest on €1.

Hence, in US dollars, his instantaneous profit is

$$
\begin{aligned}
&dx(t) + r_{\text{€}}dt \times x(t+dt) \\
= \ &dx(t) + r_{\text{€}}dt \times [x(t) + dx(t)] \\
= \ &dx(t) + x(t)r_{\text{€}}dt, \qquad\qquad \text{if } dt \times dx(t) = 0.
\end{aligned}
$$

Under the risk-neutral probability measure, the expectation of the instantaneous rate of return is the risk-free interest rate. Hence,

$$E[dx(t) + x(t)r_\epsilon dt \mid x(t)] = x(t) \times (rdt),$$

from which we obtain

$$E\left[\left.\frac{dx(t)}{x(t)}\right| x(t)\right] = (r - r_\epsilon)dt.$$

Furthermore, we now see that $\{Z(t)\}$ is a (standard) Brownian motion under the dollar-investor's risk-neutral probability measure.

By similar reasoning, we would expect

$$\frac{dy(t)}{y(t)} = (r_\epsilon - r)dt + \omega dZ_\epsilon(t),$$

where $\{Z_\epsilon(t)\}$ is a (standard) Brownian motion under the euro-investor's risk-neutral probability measure and ω is a constant. It follows from (E) that $\omega = -\sigma$ and

$$Z_\epsilon(t) = Z(t) - \sigma t.$$

Let W be a contingent claim in dollars payable at time t. Then, its time-0 price in dollars is

$$E[e^{-rt} W],$$

where the expectation is taken with respect to the dollar-investor's risk-neutral probability measure. Alternatively, let us calculate the price by the following four steps:

Step 1: We convert the time-t payoff to euros,

$$y(t)W.$$

Step 2: We discount the amount back to time 0 using the euro-denominated risk-free interest rate,

$$\exp(-r_\epsilon t)\, y(t)W.$$

Step 3: We take expectation with respect to the euro-investor's risk-neutral probability measure to obtain the contingent claim's time-0 price in euros,

$$E_\epsilon[\exp(-r_\epsilon t)\, y(t)W].$$

Here, E_ϵ signifies that the expectation is taken with respect to the euro-investor's risk-neutral probability measure.

Step 4: We convert the price in euros to a price in dollars using the time-0 exchange rate $x(0)$.

We now verify that both methods give the same price, i.e., we check that the following formula holds:

$$x(0)E_\epsilon[\exp(-r_\epsilon t)\, y(t)W] = E[e^{-rt} W].$$

This we do by using Girsanov's Theorem (McDonald 2006, p. 662). It follows from $Z_\epsilon(t) = Z(t) - \sigma t$ and footnote 9 on page 662 that

$$E_\epsilon[y(t)W] = E[\zeta(t)y(t)W],$$

where

$$\zeta(t) \;=\; \exp[-(-\sigma)Z(t) - \tfrac{1}{2}(-\sigma)^2 t] \;=\; \exp[\sigma Z(t) - \tfrac{1}{2}\sigma^2 t].$$

Because

$$y(t) \;=\; y(0)\exp[(r_{\euro} - r + \tfrac{1}{2}\sigma^2)t \,-\, \sigma Z(t)],$$

we see that

$$\exp(-r_{\euro}t)y(t)\zeta(t) \;=\; y(0)\exp(-rt).$$

Since $x(0)y(0) = 1$, we indeed have the identity

$$x(0)\mathrm{E}_{\euro}[\exp(-r_{\euro}t)\,y(t)W] \;=\; \mathrm{E}[e^{-rt}\,W].$$

If W is the payoff of a call option on euros,

$$W \;=\; [x(t) - K]_+,$$

then

$$x(0)\mathrm{E}_{\euro}[\exp(-r_{\euro}t)\,y(t)W] \;=\; \mathrm{E}[e^{-rt}\,W]$$

is a special case of identity (9.7) on page 292. A derivation of (9.7) is as follows. It is **not** necessary to assume that the exchange rate follows a geometric Brownian motion. Also, both risk-free interest rates can be stochastic.

The payoff of a t-year K-strike dollar-dominated call option on euros is

$$\begin{aligned}
&\$[x(t) - K]_+ \\
&= [\$x(t) - \$K]_+ \\
&= [\euro 1 \,-\, \$K]_+ \\
&= [\euro 1 \,-\, \euro y(t)K]_+ \\
&= K \times \euro[1/K \,-\, y(t)]_+,
\end{aligned}$$

which is K times the payoff of a t-year $(1/K)$-strike euro-dominated put option on dollars. Let $C_{\$}(x(0), K, t)$ denote the time-0 price of a t-year K-strike dollar-dominated call option on euros, and let $P_{\euro}(y(0), H, t)$ denote the time-0 price of a t-year H-strike euro-dominated put option on dollars. It follows from the time-t identity
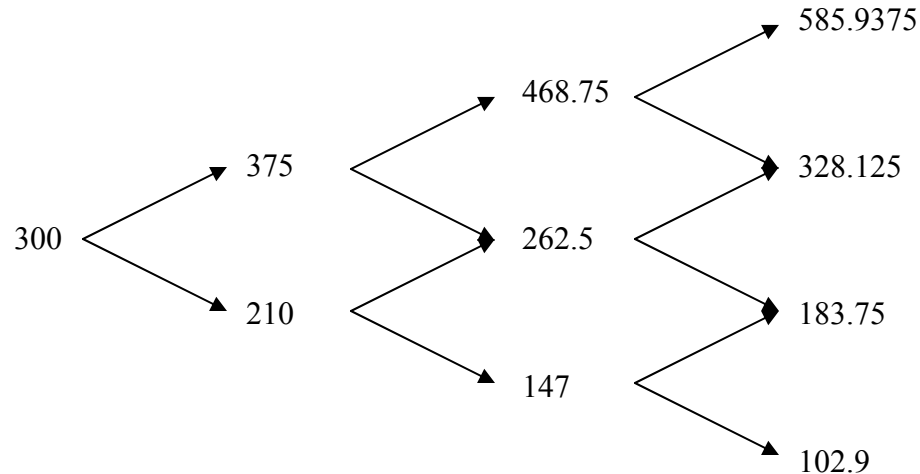
$$\$[x(t) - K]_+ \;=\; K \times \euro[1/K \,-\, y(t)]_+$$

that we have the time-0 identity

$$\begin{aligned}
\$\,C_{\$}(x(0), K, t) &= K \times \euro\, P_{\euro}(y(0), 1/K, t) \\
&= \$\,x(0) \times K \times P_{\euro}(y(0), 1/K, t) \\
&= \$\,x(0) \times K \times P_{\euro}(1/x(0), 1/K, t),
\end{aligned}$$

which is formula (9.7) on page 292.

For Questions **44** and **45**, consider the following three-period binomial tree model for a stock that pays dividends continuously at a rate proportional to its price. The length of each period is 1 year, the continuously compounded risk-free interest rate is 10%, and the continuous dividend yield on the stock is 6.5%.

44. Calculate the price of a 3-year at-the-money American put option on the stock.

(A)   15.86

(B)   27.40

(C)   32.60

(D)   39.73

(E)   57.49

45. Approximate the value of gamma at time 0 for the 3-year at-the-money American put on the stock using the method in Appendix 13.B of McDonald (2006).

(A)   0.0038

(B)   0.0041

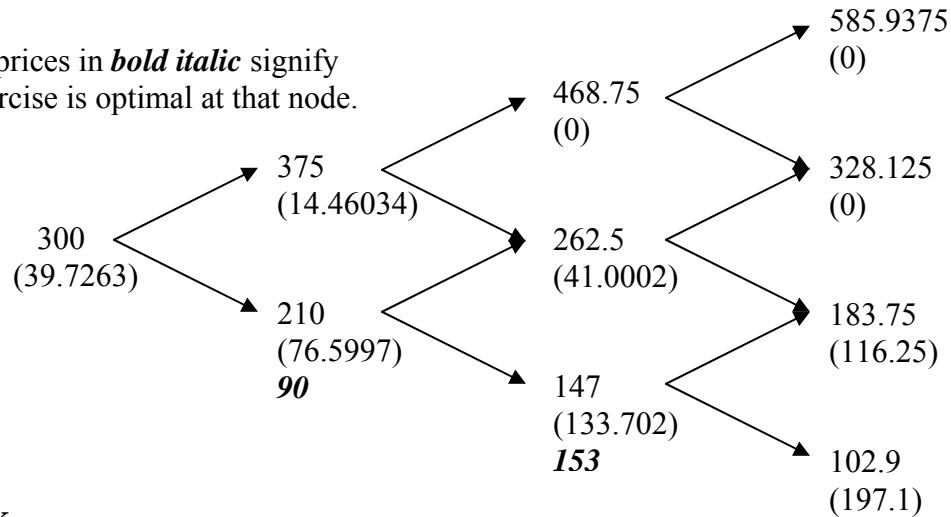(C)   0.0044

(D)   0.0047

(E)   0.0050

Solution to (44)   Answer: (D)

By formula (10.5), the risk-neutral probability of an up move is
$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{S_0 e^{(r-\delta)h} - S_d}{S_u - S_d} = \frac{300 e^{(0.1-0.065)\times 1} - 210}{375 - 210} = 0.61022.$$

Option prices in **bold italic** signify
that exercise is optimal at that node.

```
                                                      585.9375
                                                       (0)
                                       468.75
                                       (0)
                           375                        328.125
                           (14.46034)                  (0)
          300                          262.5
          (39.7263)                    (41.0002)
                           210                        183.75
                           (76.5997)                  (116.25)
                           90          147
                                       (133.702)
                                       153            102.9
                                                      (197.1)
```

Remark

If the put option is European, not American, then the simplest method is to use the
binomial formula [p. 358, (11.17); p. 618, (19.1)]:

$$e^{-r(3h)}\left[\binom{3}{3}(1-p^*)^3(300-102.9) + \binom{3}{2}p^*(1-p^*)^2(300-183.75) + 0 + 0\right]$$
$$= e^{-r(3h)}(1-p^*)^2[(1-p^*)\times 197.1 + 3\times p^*\times 116.25)]$$
$$= e^{-r(3h)}(1-p^*)^2(197.1 + 151.65 p^*)$$
$$= e^{-0.1\times 3}\times 0.38978^2\times 289.63951 = 32.5997$$

Solution to (45)   Answer: (C)

Formula (13.16) is $\Gamma = \dfrac{\Delta_u - \Delta_d}{S_u - S_d}$. By formula (13.15) (or (10.1)), $\Delta = e^{-\delta h}\dfrac{C_u - C_d}{S(u-d)}$.

$$\Delta_u = e^{-\delta h}\frac{P_{uu} - P_{ud}}{S_{uu} - S_{ud}} = e^{-0.065\times 1}\frac{0 - 41.0002}{468.75 - 262.5} = -0.186279$$

$$\Delta_d = e^{-\delta h}\frac{P_{ud} - P_{dd}}{S_{ud} - S_{dd}} = e^{-0.065\times 1}\frac{41.0002 - 153}{262.5 - 147} = -0.908670$$

Hence,
$$\Gamma = \frac{-0.186279 + 0.908670}{375 - 210} = 0.004378$$

Remark: This is an approximation, because we wish to know gamma at time 0, not at
time 1, and at the stock price $S_0 = 300$.

46. You are to price options on a futures contract. The movements of the futures price are modeled by a binomial tree. You are given:

(i) Each period is 6 months.

(ii) $u/d = 4/3$, where u is one plus the rate of gain on the futures price if it goes up, and d is one plus the rate of loss if it goes down.

(iii) The risk-neutral probability of an up move is 1/3.

(iv) The initial futures price is 80.

(v) The continuously compounded risk-free interest rate is 5%.

Let C_I be the price of a 1-year 85-strike European call option on the futures contract, and C_{II} be the price of an otherwise identical American call option.

Determine $C_{II} - C_I$.

(A) 0
(B) 0.022
(C) 0.044
(D) 0.066
(E) 0.088

Solution to (46)   Answer: (E)

By formula (10.14), the risk-neutral probability of an up move is
$$p^* = \frac{1-d}{u-d} = \frac{1/d - 1}{u/d - 1}.$$
Substituting $p^* = 1/3$ and $u/d = 4/3$, we have
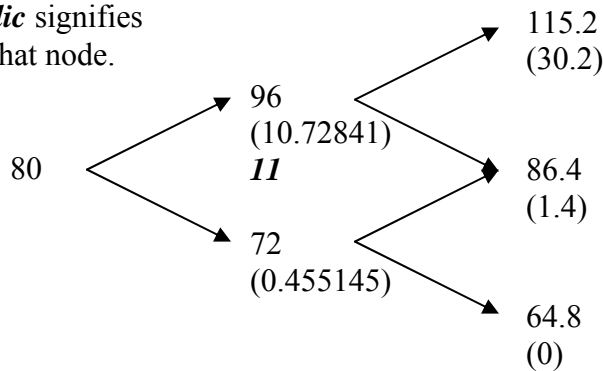$$\frac{1}{3} = \frac{1/d - 1}{4/3 - 1}.$$
Hence, $d = 0.9$ and $u = (4/3) \times d = 1.2$.

The two-period binomial tree for the futures price and prices of European and American options at $t = 0.5$ and $t = 1$ is given below. The calculation of the European option prices at $t = 0.5$ is given by
$$e^{-0.05 \times 0.5}[30.2 p^* + 1.4(1 - p^*)] = 10.72841$$
$$e^{-0.05 \times 0.5}[1.4 p^* + 0 \times (1 - p^*)] = 0.455145$$

An option price in **_bold italic_** signifies that exercise is optimal at that node.


```
                                              115.2
                                              (30.2)
                          96
                          (10.72841)
        80                11                  86.4
                                              (1.4)
                          72
                          (0.455145)
                                              64.8
                                              (0)
```

Thus, $C_{II} - C_I = e^{-0.05 \times 0.5} \times (11 - 10.72841) \times p^* = 0.088$.

Remarks:

(i) $C_I = e^{-0.05 \times 0.5}[10.72841 p^* + 0.455145(1 - p^*)] = 3.78378$.

  $C_{II} = e^{-0.05 \times 0.5}[11 p^* + 0.455145(1 - p^*)] = 3.87207$.

(ii) A futures price can be treated like a stock with $\delta = r$. With this observation, we can obtain (10.14) from (10.5),
$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(r-r)h} - d}{u - d} = \frac{1-d}{u-d}.$$
Another application is the determination of the price sensitivity of a futures option with respect to a change in the futures price. We learn from page 317 that the price sensitivity of a stock option with respect to a change in the stock price is $e^{-\delta h} \frac{C_u - C_d}{S(u - d)}$. Changing δ to r and S to F yields $e^{-rh} \frac{C_u - C_d}{F(u - d)}$, which is the same as the expression $e^{-rh}\Delta$ given in footnote 7 on page 333.

47. Several months ago, an investor sold 100 units of a one-year European call option on a nondividend-paying stock. She immediately delta-hedged the commitment with shares of the stock, but has not ever re-balanced her portfolio. She now decides to close out all positions.

You are given the following information:

(i) The risk-free interest rate is constant.

(ii)

	Several months ago	Now
Stock price	$40.00	$50.00
Call option price	$ 8.88	$14.42
Put option price	$ 1.63	$ 0.26
Call option delta	0.794	

The put option in the table above is a European option on the same stock and with the same strike price and expiration date as the call option.

Calculate her profit.

(A) $11

(B) $24

(C) $126

(D) $217

(E) $240

Solution to (47)   Answer: (B)

Let the date several months ago be 0. Let the current date be t.

Delta-hedging at time 0 means that the investor's cash position at time 0 was
$$100[C(0) - \Delta_C(0)S(0)].$$
After closing out all positions at time t, her profit is
$$100\{[C(0) - \Delta_C(0)S(0)]e^{rt} - [C(t) - \Delta_C(0)S(t)]\}.$$

To find the accumulation factor e^{rt}, we can use put-call parity:
$$C(0) - P(0) = S(0) - Ke^{-rT},$$
$$C(t) - P(t) = S(t) - Ke^{-r(T-t)},$$
where T is the option expiration date. Then,
$$e^{rt} = \frac{S(t) - C(t) + P(t)}{S(0) - C(0) + P(0)} = \frac{50 - 14.42 + 0.26}{40 - 8.88 + 1.63} = \frac{35.84}{32.75} = 1.0943511.$$

Thus, her profit is
$$100\{[C(0) - \Delta_C(0)S(0)]e^{rt} - [C(t) - \Delta_C(0)S(t)]\}$$
$$= 100\{[8.88 - 0.794 \times 40] \times 1.09435 - [14.42 - 0.794 \times 50]\}$$
$$= 24.13 \approx 24$$

Alternative Solution: Consider profit as the sum of (i) capital gain and (ii) interest:

  (i)   capital gain $= 100\{[C(0) - C(t)] - \Delta_C(0)[S(0) - S(t)]\}$
  (ii)   interest $= 100[C(0) - \Delta_C(0)S(0)](e^{rt} - 1)$.

Now,
$$\text{capital gain} = 100\{[C(0) - C(t)] - \Delta_C(0)[S(0) - S(t)]\}$$
$$= 100\{[8.88 - 14.42] - 0.794[40 - 50]\}$$
$$= 100\{-5.54 + 7.94\} = 240.00.$$
To determine the amount of interest, we first calculate her cash position at time 0:
$$100[C(0) - \Delta_C(0)S(0)] = 100[8.88 - 40 \times 0.794]$$
$$= 100[8.88 - 31.76] = -2288.00.$$
Hence,
$$\text{interest} = -2288 \times (1.09435 - 1) = -215.87.$$
Thus, the investor's profit is $240.00 - 215.87 = 24.13 \approx 24$.

Third Solution: Use the table format in Section 13.3 of McDonald (2006).

Position	Cost at time 0	Value at time t
Short 100 calls	$-100 \times 8.88 = -888$	$-100 \times 14.42 = -1442$
100Δ shares of stock	$100 \times 0.794 \times 40 = 3176$	$100 \times 0.794 \times 50 = 3970$
Borrowing	$3176 - 888 = 2288$	$2288e^{rt} = 2503.8753$
Overall	0	24.13

Remark: The problem can still be solved if the short-rate is **deterministic** (but not necessarily constant). Then, the accumulation factor e^{rt} is replaced by $\exp[\int_0^t r(s)\mathrm{d}s]$, which can be determined using the put-call parity formulas

$$C(0) - P(0) = S(0) - K\exp[-\int_0^T r(s)\mathrm{d}s],$$

$$C(t) - P(t) = S(t) - K\exp[-\int_t^T r(s)\mathrm{d}s].$$

If interest rates are stochastic, the problem as stated cannot be solved.

48. The prices of two nondividend-paying stocks are governed by the following stochastic differential equations:

$$\frac{dS_1(t)}{S_1(t)} = 0.06dt + 0.02dZ(t),$$

$$\frac{dS_2(t)}{S_2(t)} = 0.03dt + kdZ(t),$$

where $Z(t)$ is a standard Brownian motion and k is a constant.

The current stock prices are $S_1(0) = 100$ and $S_2(0) = 50$.

The continuously compounded risk-free interest rate is 4%.

You now want to construct a zero-investment, risk-free portfolio with the two stocks and risk-free bonds.

If there is exactly one share of Stock 1 in the portfolio, determine the number of shares of Stock 2 that you are now to buy. (A negative number means shorting Stock 2.)

(A)   – 4

(B)   – 2

(C)   – 1

(D)   1

(E)   4

Solution to (48)   Answer: (E)

The problem is a variation of Exercise 20.12 where one asset is perfectly *negatively* correlated with another.

The no-arbitrage argument in Section 20.4 "The Sharpe Ratio" shows that
$$\frac{0.06 - 0.04}{0.02} = \frac{0.03 - 0.04}{k}$$
or $k = -0.01$, and that the current number of shares of Stock 2 in the hedged portfolio is
$$-\frac{\sigma_1 \times S_1(0)}{k \times S_2(0)} = -\frac{0.02 \times 100}{(-0.01) \times 50} = 4,$$
which means buying four shares of Stock 2.

Alternative Solution: Construct the zero-investment, risk-free portfolio by following formula (21.7) or formula (24.4):
$$I(t) = S_1(t) + N(t)S_2(t) + W(t),$$
where $N(t)$ is the number of shares of Stock 2 in the portfolio at time t and $W(t)$ is the amount of short-term bonds so that $I(t) = 0$, i.e.,
$$W(t) = -[S_1(t) + N(t)S_2(t)].$$
Our goal is to find $N(0)$. Now, the instantaneous change in the portfolio value is
$$\begin{aligned}
dI(t) &= dS_1(t) + N(t)dS_2(t) + W(t)r dt \\
&= S_1(t)[0.06dt + 0.02dZ(t)] + N(t)S_2(t)[0.03dt + kdZ(t)] + 0.04W(t)dt \\
&= \alpha(t)dt + \beta(t)dZ(t),
\end{aligned}$$
where
$$\begin{aligned}
\alpha(t) &= 0.06S_1(t) + 0.03N(t)S_2(t) - 0.04[S_1(t) + N(t)S_2(t)] \\
&= 0.02S_1(t) - 0.01N(t)S_2(t),
\end{aligned}$$
and
$$\beta(t) = 0.02S_1(t) + kN(t)S_2(t).$$

The portfolio is risk-free means that $N(t)$ is such that $\beta(t) = 0$. Since $I(t) = 0$, the no-arbitrage condition and the risk-free condition mean that we must also have $\alpha(t) = 0$, or
$$N(t) = \frac{0.02S_1(t)}{0.01S_2(t)}.$$
In particular,
$$N(0) = \frac{0.02S_1(0)}{0.01S_2(0)} = \frac{2}{0.5} = 4.$$

Remark: Equation (21.20) on page 687 of McDonald (2006) should be the same as (12.9) on page 393,
$$\sigma_{\text{option}} = |\Omega| \times \sigma.$$
Thus, (21.21) should be changed to
$$\frac{\alpha - r}{\sigma} = \text{sign}(\Omega) \times \frac{\alpha_{\text{option}} - r}{\sigma_{\text{option}}}.$$

49. You use the usual method in McDonald and the following information to construct a one-period binomial tree for modeling the price movements of a nondividend-paying stock. (The tree is sometimes called a forward tree).

(i) The period is 3 months.

(ii) The initial stock price is $100.

(iii) The stock's volatility is 30%.

(iv) The continuously compounded risk-free interest rate is 4%.

At the beginning of the period, an investor owns an American put option on the stock. The option expires at the end of the period.

Determine the smallest integer-valued strike price for which an investor will exercise the put option at the beginning of the period.

(A)   114

(B)   115

(C)   116

(D)   117

(E)   118

Solution to (49)   Answer: (B)

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{rh+\sigma\sqrt{h}} = e^{(0.04/4)+(0.3/2)} = e^{0.16} = 1.173511$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{rh-\sigma\sqrt{h}} = e^{(0.04/4)-(0.3/2)} = e^{-0.14} = 0.869358$$
$$S = \text{initial stock price} = 100$$

The problem is to find the smallest integer K satisfying
$$K - S > e^{-rh}[p^* \times \text{Max}(K - Su, 0) + (1 - p^*) \times \text{Max}(K - Sd, 0)]. \tag{1}$$

Because the RHS of (1) is nonnegative (the payoff of an option is nonnegative), we have the condition
$$K - S > 0. \tag{2}$$

As $d < 1$, it follows from condition (2) that
$$\text{Max}(K - Sd, 0) = K - Sd,$$
and inequality (1) becomes
$$K - S > e^{-rh}[p^* \times \text{Max}(K - Su, 0) + (1 - p^*) \times (K - Sd)]. \tag{3}$$

If $K \geq Su$, the right-hand side of (3) is
$$e^{-rh}[p^* \times (K - Su) + (1 - p^*) \times (K - Sd)]$$
$$= e^{-rh}K - e^{-\delta h}S$$
$$= e^{-rh}K - S,$$
because the stock pays no dividends. Thus, if $K \geq Su$, inequality (3) always holds, and the put option is exercised early.

We now investigate whether there is any K, $S < K < Su$, such that inequality (3) holds. If $Su > K$, then $\text{Max}(K - Su, 0) = 0$ and inequality (3) simplifies as
$$K - S > e^{-rh} \times (1 - p^*) \times (K - Sd),$$
or
$$K > \frac{1 - e^{-rh}(1 - p^*)d}{1 - e^{-rh}(1 - p^*)} S. \tag{4}$$

The fraction $\dfrac{1 - e^{-rh}(1 - p^*)d}{1 - e^{-rh}(1 - p^*)}$ can be simplified as follows, but this step is not necessary. In McDonald's forward-tree model,
$$1 - p^* = p^* \times e^{\sigma\sqrt{h}},$$
from which we obtain
$$1 - p^* = \frac{1}{1 + e^{-\sigma\sqrt{h}}}.$$

Hence,
$$\frac{1-e^{-rh}(1-p*)d}{1-e^{-rh}(1-p*)} = \frac{1+e^{-\sigma\sqrt{h}}-e^{-rh}d}{1+e^{-\sigma\sqrt{h}}-e^{-rh}}$$

$$= \frac{1+e^{-\sigma\sqrt{h}}-e^{-\sigma\sqrt{h}}}{1+e^{-\sigma\sqrt{h}}-e^{-rh}} \qquad \text{because } \delta = 0$$

$$= \frac{1}{1+e^{-\sigma\sqrt{h}}-e^{-rh}}.$$

Therefore, inequality (4) becomes
$$K > \frac{1}{1+e^{-\sigma\sqrt{h}}-e^{-rh}} S$$

$$= \frac{1}{1+e^{-0.15}-e^{-0.01}} S = 1.148556 \times 100 = 114.8556.$$

Thus, the answer to the problem is $\lceil 114.8556 \rceil = 115$, which is (B).

Alternative Solution:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{rh+\sigma\sqrt{h}} = e^{(0.04/4)+(0.3/2)} = e^{0.16} = 1.173511$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{rh-\sigma\sqrt{h}} = e^{(0.04/4)-(0.3/2)} = e^{-0.14} = 0.869358$$

$$S = \text{initial stock price} = 100$$

$$p* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.3/2}} = \frac{1}{1+e^{0.15}} = \frac{1}{1+1.1618} = 0.46257.$$

Then, inequality (1) is
$$K - 100 > e^{-0.01}[0.4626 \times (K-117.35)_+ + 0.5374 \times (K-86.94)_+], \qquad (5)$$
and we check three cases: $K \le 86.94$, $K \ge 117.35$, and $86.94 < K < 117.35$.

For $K \le 86.94$, inequality (5) cannot hold, because its LHS < 0 and its RHS = 0.
For $K \ge 117.35$, (5) always holds, because its LHS = $K - 100$ while
its RHS = $e^{-0.01}K - 100$.
For $86.94 < K < 117.35$, inequality (5) becomes
$$K - 100 > e^{-0.01} \times 0.5374 \times (K-86.94),$$
or

$$K > \frac{100 - e^{-0.01} \times 0.5374 \times 86.94}{1 - e^{-0.01} \times 0.5374} = 114.85.$$

Third Solution: Use the method of trial and error. For $K = 114, 115, \ldots$, check whether inequality (5) holds.

Remark: An American call option on a nondividend-paying stock is never exercised early. This problem shows that the corresponding statement for American puts is not true.

50. Assume the Black-Scholes framework.

You are given the following information for a stock that pays dividends continuously at a rate proportional to its price.

(i)   The current stock price is 0.25.

(ii)   The stock's volatility is 0.35.

(iii) The continuously compounded expected rate of stock-price appreciation is 15%.

Calculate the upper limit of the 90% lognormal confidence interval for the price of the stock in 6 months.

(A)   0.393

(B)   0.425

(C)   0.451

(D)   0.486

(E)   0.529

Solution to (50)   Answer: (A)

This problem is a modification of #4 in the May 2007 Exam C.

The conditions given are:

   (i) $S_0 = 0.25$,

   (ii) $\sigma = 0.35$,

   (iii) $\alpha - \delta = 0.15$.

We are to seek the number $S_{0.5}^{U}$ such that $\Pr(S_{0.5} < S_{0.5}^{U}) = 0.95$.

The random variable $\ln(S_{0.5} / 0.25)$ is normally distributed with

$$\text{mean} = (0.15 - \tfrac{1}{2} \times 0.35^2) \times 0.5 = 0.044375,$$
$$\text{standard deviation} = 0.35 \times \sqrt{0.5} = 0.24749.$$

Because $N^{-1}(0.95) = 1.645$, we have
$$0.044375 + 0.24749 N^{-1}(0.95) = 0.4515.$$

Thus,

$$S_{0.5}^{U} = 0.25 \times e^{0.4515} = 0.3927.$$

Remark The term "confidence interval" as used in Section 18.4 seems incorrect, because S_t is a random variable, not an unknown, but constant, parameter. The expression
$$\Pr(S_t^{L} < S_t < S_t^{U}) = 1 - p$$
gives the probability that the random variable S_t is between S_t^{L} and S_t^{U}, not the "confidence" for S_t to be between S_t^{L} and S_t^{U}.

51. Assume the Black-Scholes framework.

The price of a nondividend-paying stock in seven consecutive months is:

Month	Price
1	54
2	56
3	48
4	55
5	60
6	58
7	62

Estimate the continuously compounded expected rate of return on the stock.

(A) Less than 0.28

(B) At least 0.28, but less than 0.29

(C) At least 0.29, but less than 0.30

(D) At least 0.30, but less than 0.31

(E) At least 0.31

Solution to (51)   Answer: (E)

This problem is a modification of #34 in the May 2007 Exam C. Note that you are given monthly prices, but you are asked to find an annual rate.

It is assumed that the stock price process is given by

$$\frac{dS(t)}{S(t)} = \alpha\,dt + \sigma\,dZ(t), \qquad\qquad t \geq 0.$$

We are to estimate α, using observed values of $S(jh)$, $j = 0, 1, 2, .. , n$, where $h = 1/12$ and $n = 6$. The solution to the stochastic differential equation is

$$S(t) = S(0)\exp[(\alpha - \tfrac{1}{2}\sigma^2)t + \sigma Z(t)].$$

Thus, $\ln[S((j+1)h)/S(jh)]$, $j = 0, 1, 2, \ldots$, are i.i.d. normal random variables with mean $(\alpha - \tfrac{1}{2}\sigma^2)h$ and variance $\sigma^2 h$.

Let $\{r_j\}$ denote the observed continuously compounded monthly returns:

$$r_1 = \ln(56/54) = 0.03637,$$
$$r_2 = \ln(48/56) = -0.15415,$$
$$r_3 = \ln(55/48) = 0.13613,$$
$$r_4 = \ln(60/55) = 0.08701,$$
$$r_5 = \ln(58/60) = -0.03390,$$
$$r_6 = \ln(62/58) = 0.06669.$$

The sample mean is

$$\bar{r} = \frac{1}{n}\sum_{j=1}^{n} r_j = \frac{1}{n}\ln\frac{S(t_{nh})}{S(t_0)} = \frac{1}{6}\ln\frac{62}{54} = 0.023025.$$

The (unbiased) sample variance is

$$\frac{1}{n-1}\sum_{j=1}^{n}(r_j - \bar{r})^2 = \frac{1}{n-1}\left[\sum_{j=1}^{n}(r_j)^2 - n\bar{r}^2\right] = \frac{1}{5}\left[\sum_{j=1}^{6}(r_j)^2 - 6\bar{r}^2\right] = 0.01071.$$

Thus, $\alpha = (\alpha - \tfrac{1}{2}\sigma^2) + \tfrac{1}{2}\sigma^2$ is estimated by

$$(0.023025 + \tfrac{1}{2} \times 0.01071) \times 12 = 0.3405.$$

Remarks:

(i) Let $T = nh$. Then the estimator of $\alpha - \frac{1}{2}\sigma^2$ is

$$\frac{\bar{r}}{h} = \frac{1}{nh}\ln\frac{S(T)}{S(0)} = \frac{\ln[S(T)] - \ln[S(0)]}{T - 0}.$$

This is a special case of the result that the drift of an arithmetic Brownian motion is estimated by the slope of the straight line joining its first and last observed values. Observed values of the arithmetic Brownian motion in between are not used.

(ii) An (unbiased) estimator of σ^2 is

$$\frac{1}{h}\frac{1}{n-1}\left[\sum_{j=1}^{n}(r_j)^2 - n\bar{r}^2\right] = \frac{1}{T}\left\{\frac{n}{n-1}\sum_{j=1}^{n}(r_j)^2 - \frac{1}{n-1}\left[\ln\frac{S(T)}{S(0)}\right]^2\right\}$$

$$\approx \frac{1}{T}\frac{n}{n-1}\sum_{j=1}^{n}(r_j)^2 \qquad\qquad \text{for large } n \text{ (small } h\text{)}$$

$$= \frac{1}{T}\frac{n}{n-1}\sum_{j=1}^{n}\{\ln[S(jT/n)/S((j-1)T/n)]\}^2,$$

which can be found in footnote 9 on page 756 of McDonald (2006). It is equivalent to formula (23.2) on page 744 of McDonald (2006), which is

$$\hat{\sigma}_H^2 = \frac{1}{h}\frac{1}{n-1}\sum_{j=1}^{n}\{\ln[S(jT/n)/S((j-1)T/n)]\}^2.$$

(iii) An important result (McDonald 2006, p. 653, p. 755) is: With probability 1,

$$\lim_{n\to\infty}\sum_{j=1}^{n}\{\ln[S(jT/n)/S((j-1)T/n)]\}^2 = \sigma^2 T,$$

showing that the exact value of σ can be obtained by means of a single sample path of the stock price. Here is an implication of this result. Suppose that an actuary uses a so-called regime-switching model to model the price of a stock (or stock index), with each regime being characterized by a different σ. In such a model, the current regime can be determined by this formula. If the price of the stock can be observed over a time interval, no matter how short the time interval is, then σ is revealed immediately by determining the *quadratic variation* of the logarithm of the stock price.

52. The price of a stock is to be estimated using simulation. It is known that:

(i) The time-t stock price, S_t, follows the lognormal distribution:
$$\ln\left(\frac{S_t}{S_0}\right) \sim N((\alpha - \tfrac{1}{2}\sigma^2)t, \sigma^2 t)$$

(ii) $S_0 = 50$, $\alpha = 0.15$, and $\sigma = 0.30$.

The following are three uniform $(0, 1)$ random numbers

$$0.9830 \qquad 0.0384 \qquad 0.7794$$

Use each of these three numbers to simulate a time-2 stock price.

Calculate the mean of the three simulated prices.

(A)　Less than 75

(B)　At least 75, but less than 85

(C)　At least 85, but less than 95

(D)　At least 95, but less than 115

(E)　At least 115

Solution to (52)   Answer: (C)

This problem is a modification of #19 in the May 2007 Exam C.

> $U \sim$ Uniform $(0, 1)$
>
> $\Rightarrow N^{-1}(U) \sim \mathcal{N}(0, 1)$
>
> $\Rightarrow a + bN^{-1}(U) \sim \mathcal{N}(a, b^2)$

The random variable $\ln(S_2 / 50)$ has a normal distribution with mean $(0.15 - \frac{1}{2} \times 0.3^2) \times 2 = 0.21$ and variance $0.3^2 \times 2 = 0.18$, and thus a standard deviation of 0.4243.

The three uniform random numbers become the following three values from the standard normal: 2.12, −1.77, 0.77. Upon multiplying each by the standard deviation of 0.4243 and adding the mean of 0.21, the resulting normal values are 1.109, −0.541, and 0.537. The simulated stock prices are obtained by exponentiating these numbers and multiplying by 50. This yields 151.57, 29.11, and 85.54. The average of these three numbers is 88.74.

53. Assume the Black-Scholes framework. For a European put option and a European gap call option on a stock, you are given:

(i)   The expiry date for both options is T.

(ii)   The put option has a strike price of 40.

(iii)   The gap call option has strike price 45 and payment trigger 40.

(iv)   The time-0 gamma of the put option is 0.07.

(v)   The time-0 gamma of the gap call option is 0.08.

Consider a European cash-or-nothing call option that pays 1000 at time T if the stock price at that time is higher than 40.

Find the time-0 gamma of the cash-or-nothing call option.

(A)   −5

(B)   −2

(C)   2

(D)   5

(E)   8

Solution to (53)   Answer: (B)

Let I[.] be the *indicator function*, i.e., $I[A] = 1$ if the event A is true, and $I[A] = 0$ if the event A is false. Let K_1 be the strike price and K_2 be the payment trigger of the gap call option. The payoff of the gap call option is

$$[S(T) - K_1] \times I[S(T) > K_2] = \underbrace{[S(T) - K_2] \times I[S(T) > K_2]}_{\substack{\text{payoff of} \\ \text{a } K_2\text{-strike call}}} + \underbrace{(K_2 - K_1) \times I[S(T) > K_2]}_{\substack{(K_2 - K_1) \text{ times the payoff of} \\ \text{a cash-or-nothing call} \\ \text{that pays \$1 if } S(T) > K_2}}.$$

Because differentiation is a linear operation, each Greek (except for omega or elasticity) of a portfolio is the sum of the corresponding Greeks for the components of the portfolio (McDonald 2006, page 395). Thus,

Gap call gamma = Call gamma + $(K_2 - K_1) \times$ Cash-or-nothing call gamma

As pointed out on line 12 of page 384 of McDonald (2006), call gamma equals put gamma. (To see this, differentiate the put-call parity formula twice with respect to S.)

Because $K_2 - K_1 = 40 - 45 = -5$, call gamma = put gamma = 0.07, and gap call gamma = 0.08, we have

$$\text{Cash-or-nothing call gamma} = \frac{0.08 - 0.07}{-5} = -0.002$$

Hence the answer is $1000 \times (-0.002) = -2$.

Remark: Another decomposition of the payoff of the gap call option is the following:

$$[S(T) - K_1] \times I[S(T) > K_2] = \underbrace{S(T) \times I[S(T) > K_2]}_{\substack{\text{payoff of an} \\ \text{asset-or-nothing call}}} - \underbrace{K_1 \times I[S(T) > K_2]}_{\substack{K_1 \text{ times the payoff of} \\ \text{a cash-or-nothing call} \\ \text{that pays \$1 if } S(T) > K_2}}.$$

See page 707 of McDonald (2006). Such a decomposition, however, is not useful here.

54. Assume the Black-Scholes framework. Consider two nondividend-paying stocks whose time-t prices are denoted by $S_1(t)$ and $S_2(t)$, respectively.

You are given:

(i) $S_1(0) = 10$ and $S_2(0) = 20$.

(ii)   Stock 1's volatility is 0.18.

(iii) Stock 2's volatility is 0.25.

(iv)   The correlation between the continuously compounded returns of the two stocks is –0.40.

(v)   The continuously compounded risk-free interest rate is 5%.

(vi)   A one-year European option with payoff $\max\{\min[2S_1(1), S_2(1)] - 17, 0\}$ has a current (time-0) price of 1.632.

Consider a European option that gives its holder the right to sell either two shares of Stock 1 or one share of Stock 2 at a price of 17 one year from now.

Calculate the current (time-0) price of this option.

(A)   0.66

(B)   1.12

(C)   1.49

(D)   5.18

(E)   7.86

Solution to (54)   Answer: (A)

At the option-exercise date, the option holder will sell two shares of Stock 1 or one share of Stock 2, depending on which trade is of lower cost. Thus, the time-1 payoff of the option is

$$\max\{17 - \min[2S_1(1), S_2(1)], 0\},$$

which is the payoff of a 17-strike put on $\min[2S_1(1), S_2(1)]$. Define

$$M(T) = \min[2S_1(T), S_2(T)].$$

Consider put-call parity with respect to $M(T)$:

$$c(K, T) - p(K, T) = F_{0,T}^P(M) - Ke^{-rT}.$$

Here, $K = 17$ and $T = 1$. It is given in (vi) that $c(17, 1) = 1.632$. $F_{0,1}^P(M)$ is the time-0 price of the security with time-1 payoff

$$M(1) = \min[2S_1(1), S_2(1)] = 2S_1(1) - \max[2S_1(1) - S_2(1), 0].$$

Since $\max[2S_1(1) - S_2(1), 0]$ is the payoff of an exchange option, its price can be obtained using (14.16) and (14.17):

$$\sigma = \sqrt{0.18^2 + 0.25^2 - 2(-0.4)(0.18)(0.25)} = 0.3618$$

$$d_1 = \frac{\ln[2S_1(0)/S_2(0)] + \tfrac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \tfrac{1}{2}\sigma\sqrt{T} = 0.1809 \approx 0.18, \ N(d_1) = 0.5714$$

$$d_2 = d_1 - \sigma\sqrt{T} = -\tfrac{1}{2}\sigma\sqrt{T} \approx -0.18, \ N(d_2) = 1 - 0.5714 = 0.4286$$

Price of the exchange option $= 2S_1(0)N(d_1) - S_2(0)N(d_2) = 20N(d_1) - 20N(d_2) = 2.856$

Thus,

$$F_{0,1}^P(M) = 2F_{0,1}^P(S_1) - 2.856 = 2 \times 10 - 2.856 = 17.144$$

and

$$p(17, 1) = 1.632 - 17.144 + 17e^{-0.05} = 0.6589.$$

Remarks: (i) The exchange option above is an "at-the-money" exchange option because $2S_1(0) = S_2(0)$. See also Example 14.3 of McDonald (2006).

(ii) Further discussion on exchange options can be found in Section 22.6, which is not part of the MFE/3F syllabus. Q and S in Section 22.6 correspond to $2S_1$ and S_2 in this problem.

55. Assume the Black-Scholes framework. Consider a 9-month at-the-money European put option on a futures contract. You are given:

(i) The continuously compounded risk-free interest rate is 10%.

(ii) The strike price of the option is 20.

(iii) The price of the put option is 1.625.

If three months later the futures price is 17.7, what is the price of the put option at that time?

(A) 2.09

(B) 2.25

(C) 2.45

(D) 2.66

(E) 2.83

Solution to (55)   Answer: (D)

By (12.7), the price of the put option is
$$P = e^{-rT}[KN(-d_2) - FN(-d_1)],$$

where $d_1 = \dfrac{\ln(F/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$, and $d_2 = d_1 - \sigma\sqrt{T}$.

With $F = K$, we have $\ln(F/K) = 0$, $d_1 = \dfrac{\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \frac{1}{2}\sigma\sqrt{T}$, $d_2 = -\frac{1}{2}\sigma\sqrt{T}$, and

$$P = Fe^{-rT}[N(\tfrac{1}{2}\sigma\sqrt{T}) - N(-\tfrac{1}{2}\sigma\sqrt{T})] = Fe^{-rT}[2N(\tfrac{1}{2}\sigma\sqrt{T}) - 1].$$

Putting $P = 1.6$, $r = 0.1$, $T = 0.75$, and $F = 20$, we get
$$1.625 = 20e^{-0.1\times0.75}[2N(\tfrac{1}{2}\sigma\sqrt{0.75}) - 1]$$
$$N(\tfrac{1}{2}\sigma\sqrt{0.75}) = 0.5438$$
$$\tfrac{1}{2}\sigma\sqrt{0.75} = 0.11$$
$$\sigma = 0.254$$

After 3 months, we have $F = 17.7$ and $T = 0.5$; hence
$$d_1 = \frac{\ln(F/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln(17.7/20) + \frac{1}{2}\times0.254^2\times0.5}{0.254\sqrt{0.5}} = -0.5904 \approx -0.59$$

$N(-d_1) \approx 0.7224$

$d_2 = d_1 - \sigma\sqrt{T} = -0.5904 - 0.254\sqrt{0.5} = -0.7700$

$N(-d_2) \approx 0.7794$

The put price at that time is
$$\begin{aligned}
P &= e^{-rT}[KN(-d_2) - FN(-d_1)] \\
&= e^{-0.1\times0.5}[20\times0.7794 - 17.7\times0.7224] \\
&= 2.66489
\end{aligned}$$

Remarks:

(i)   A somewhat related problem is #8 in the May 2007 MFE exam. Also see the box on page 299 and the one on page 603 of McDonald (2006).

(ii)   For European call and put options on a futures contract with the same exercise date, the call price and put price are the same if and only if both are at-the-money options. The result follows from put-call parity. See the first equation in Table 9.9 on page 305 of McDonald (2006).

(iii) The point above can be generalized. It follows from the identity

$$[S_1(T) - S_2(T)]_+ + S_2(T) = [S_2(T) - S_1(T)]_+ + S_1(T)$$

that

$$F^P_{0,T}((S_1 - S_2)_+) + F^P_{0,T}(S_2) = F^P_{0,T}((S_2 - S_1)_+) + F^P_{0,T}(S_1).$$

(See also formula 9.6 on page 287.) Note that $F^P_{0,T}((S_1 - S_2)_+)$ and

$F^P_{0,T}((S_2 - S_1)_+)$ are time-0 prices of exchange options. The two exchange options

have the same price if and only if the two prepaid forward prices, $F^P_{0,T}(S_1)$ and

$F^P_{0,T}(S_2)$, are the same.

56. Assume the Black-Scholes framework. For a stock that pays dividends continuously at a rate proportional to its price, you are given:

(i) The current stock price is 5.

(ii) The stock's volatility is 0.2.

(iii) The continuously compounded expected rate of stock-price appreciation is 5%.

Consider a 2-year arithmetic average strike option. The strike price is

$$A(2) = \frac{1}{2}[S(1) + S(2)].$$

Calculate Var[$A(2)$].

(A) 1.51

(B) 5.57

(C) 10.29

(D) 22.29

(E) 30.57

Solution to (56)   Answer: (A)

$$\mathrm{Var}[A(2)] \;=\; \left(\frac{1}{2}\right)^2 \{\mathrm{E}[(S(1)+S(2))^2] \;-\; (\mathrm{E}[S(1)+S(2)])^2\}.$$

The second expectation is easier to evaluate. By (20.29) on page 665 of McDonald (2006),
$$S(t) = S(0)\exp[(\alpha - \delta - \tfrac{1}{2}\sigma^2)t + \sigma Z(t)].$$
Thus,
$$\mathrm{E}[S(t)] = S(0)\exp[(\alpha - \delta - \tfrac{1}{2}\sigma^2)t]\times\mathrm{E}[e^{\sigma Z(t)}]$$
$$= S(0)\exp[(\alpha - \delta)t]$$
by (18.13). (See also formulas 18.21 and 18.22.) Consequently,
$$\mathrm{E}[S(1)+S(2)] = \mathrm{E}[S(1)] + \mathrm{E}[S(2)]$$
$$= 5(e^{0.05} + e^{0.1}),$$
because condition (iii) means that $\alpha - \delta = 0.05$.

We now evaluate the first expectation, $\mathrm{E}[(S(1)+S(2))^2]$. Because
$$\frac{S(t+1)}{S(t)} = \exp\{(\alpha - \delta - \tfrac{1}{2}\sigma^2) + \sigma[Z(t+1) - Z(t)]\}$$
and because $\{Z(t+1) - Z(t),\ t = 0, 1, 2, ...\}$ are i.i.d. $\mathcal{N}(0, 1)$ random variables (the second and third points at the bottom of page 650), we see that
$$\left\{\frac{S(t+1)}{S(t)},\ t = 0,\ 1,\ 2,\ ...\right\} \text{ is a sequence of i.i.d. random variables. Thus,}$$

$$\mathrm{E}[(S(1)+S(2))^2] = \mathrm{E}\left[S^2(1)\left(1 + \frac{S(2)}{S(1)}\right)^2 \right]$$

$$= S^2(0)\times\mathrm{E}\left[\left(\frac{S(1)}{S(0)}\right)^2\right]\times\mathrm{E}\left[\left(1 + \frac{S(2)}{S(1)}\right)^2\right]$$

$$= S^2(0)\times\mathrm{E}\left[\left(\frac{S(1)}{S(0)}\right)^2\right]\times\mathrm{E}\left[\left(1 + \frac{S(1)}{S(0)}\right)^2\right].$$

By the last equation on page 667, we have
$$\mathrm{E}\left[\left(\frac{S(t)}{S(0)}\right)^a\right] = e^{[a(\alpha - \delta) + \frac{1}{2}a(a-1)\sigma^2]t}.$$

(This formula can also be obtained from (18.18) and (18.13). A formula equivalent to (18.13) will be provided to candidates writing Exam MFE/3F. See the last formula on

the first page in http://www.soa.org/files/pdf/edu-2009-fall-mfe-table.pdf) With $a = 2$
and $t = 1$, the formula becomes

$$\mathrm{E}\left[\left(\frac{S(1)}{S(0)}\right)^2\right] = \exp[2 \times 0.05 + 0.2^2] = e^{0.14}.$$

Furthermore,

$$\mathrm{E}\left[\left(1 + \frac{S(1)}{S(0)}\right)^2\right] = 1 + 2\mathrm{E}\left[\frac{S(1)}{S(0)}\right] + \mathrm{E}\left[\left(\frac{S(1)}{S(0)}\right)^2\right] = 1 + 2e^{0.05} + e^{0.14}.$$

Hence,
$$\mathrm{E}[(S(1) + S(2))^2] = 5^2 \times e^{0.14} \times (1 + 2e^{0.05} + e^{0.14}) = 122.29757.$$

Finally,
$$\mathrm{Var}[A(2)] = \tfrac{1}{4} \times \{122.29757 - [5(e^{0.05} + e^{0.1})]^2\} = 1.51038.$$

Alternative Solution:

$$\mathrm{Var}[S(1) + S(2)] = \mathrm{Var}[S(1)] + \mathrm{Var}[S(2)] + 2\mathrm{Cov}[S(1), S(2)].$$

Because $S(t)$ is a lognormal random variable, the two variances can be evaluated using
the following formula, which is a consequence of (18.14) on page 595.

$$\begin{aligned}
\mathrm{Var}[S(t)] &= \mathrm{Var}[S(0)\exp[(\alpha - \delta - \tfrac{1}{2}\sigma^2)t + \sigma Z(t)] \\
&= S^2(0)\exp[2(\alpha - \delta - \tfrac{1}{2}\sigma^2)t]\mathrm{Var}[e^{\sigma Z(t)}] \\
&= S^2(0)\exp[2(\alpha - \delta - \tfrac{1}{2}\sigma^2)t]\exp(\sigma^2 t)[\exp(\sigma^2 t) - 1] \\
&= S^2(0) \, e^{2(\alpha - \delta)t} [\exp(\sigma^2 t) - 1].
\end{aligned}$$

(As a check, we can use the well-known formula for the square of the *coefficient of
variation* of a lognormal random variable. In this case, it takes the form

$$\frac{\mathrm{Var}[S(t)]}{\{\mathrm{E}[S(t)]\}^2} = e^{\sigma^2 t} - 1.$$

This matches with the results above. The coefficient of variation is in the syllabus of
Exam C/4.)

To evaluate the covariance, we can use the formula

$$\mathrm{Cov}(X, Y) = \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y].$$

In this case, however, there is a better covariance formula:

$$\mathrm{Cov}(X, Y) = \mathrm{Cov}[X, \mathrm{E}(Y \mid X)].$$

Thus,

$$
\begin{aligned}
\mathrm{Cov}[S(1), S(2)] &= \mathrm{Cov}[S(1), \mathrm{E}[S(2)|S(1)]] \\
&= \mathrm{Cov}[S(1), S(1)\mathrm{E}[S(2)/S(1)|S(1)]] \\
&= \mathrm{Cov}[S(1), S(1)\mathrm{E}[S(1)/S(0)]] \\
&= \mathrm{E}[S(1)/S(0)]\mathrm{Cov}[S(1), S(1)] \\
&= e^{\alpha - \delta}\, \mathrm{Var}[S(1)].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathrm{Var}[S(1) + S(2)] &= (1 + 2e^{\alpha - \delta})\mathrm{Var}[S(1)] + \mathrm{Var}[S(2)] \\[2mm]
&= [S(0)]^2[(1 + 2e^{\alpha - \delta})e^{2(\alpha - \delta)}(e^{\sigma^2} - 1) + e^{4(\alpha - \delta)}(e^{2\sigma^2} - 1)] \\
&= 25[(1 + 2e^{0.05})e^{0.1}(e^{0.04} - 1) + e^{0.2}(e^{0.08} - 1)] \\
&= 6.041516,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Var}[A(2)] &= \mathrm{Var}[S(1) + S(2)]/4 \\
&= 6.041516\,/\,4 \\
&= 1.510379.
\end{aligned}
$$

Remark: #37 in this set of sample questions is on determining the variance of a geometric average. It is an easier problem.

57. Michael uses the following method to simulate 8 standard normal random variates:

Step 1: Simulate 8 uniform (0, 1) random numbers $U_1, U_2, \ldots, U_8$.

Step 2: Apply the stratified sampling method to the random numbers so that U_i and U_{i+4} are transformed to random numbers V_i and V_{i+4} that are uniformly distributed over the interval $((i-1)/4, i/4)$, $i = 1, 2, 3, 4$. In each of the four quartiles, a smaller value of U results in a smaller value of V.

Step 3: Compute 8 standard normal random variates by $Z_i = N^{-1}(V_i)$, where N^{-1} is the inverse of the cumulative standard normal distribution function.

Michael draws the following 8 uniform (0, 1) random numbers:

i	1	2	3	4	5	6	7	8
U_i	0.4880	0.7894	0.8628	0.4482	0.3172	0.8944	0.5013	0.3015

Find the difference between the largest and the smallest simulated normal random variates.

(A)   0.35

(B)   0.78

(C)   1.30

(D)   1.77

(E)   2.50

Solution to (57)   Answer: (E)

The following transformation in McDonald (2006, page 632),

$$\frac{i-1+u_i}{100}, \qquad i = 1, 2, 3, \ldots, 100,$$

is now changed to

$$\frac{i-1+U_{i \text{ or } i+4}}{4}, \quad i = 1, 2, 3, 4.$$

Since the smallest Z comes from the first quartile, it must come from U_1 or U_5.

Since $U_5 < U_1$, we use U_5 to compute the smallest Z:

$$V_5 = \frac{1-1+0.3172}{4} = 0.0793,$$

$$Z_5 = N^{-1}(0.0793) = -N^{-1}(0.9207) = -1.41.$$

Since the largest Z comes from the fourth quartile, it must come from U_4 and U_8.

Since $U_4 > U_8$, we use U_4 to compute the largest Z:

$$V_4 = \frac{4-1+0.4482}{4} = 0.86205 \approx 0.8621,$$

$$Z_4 = N^{-1}(0.8621) = 1.09.$$

The difference between the largest and the smallest normal random variates is

$$Z_4 - Z_5 = 1.09 - (-1.41) = 2.50.$$

Remark:

The simulated standard normal random variates are as follows:

i	1	2	3	4	5	6	7	8
U_i	0.4880	0.7894	0.8628	0.4482	0.3172	0.8944	0.5013	0.3015
no stratified sampling	−0.030	0.804	1.093	−0.130	−0.476	1.250	0.003	−0.520
V_i	0.1220	0.4474	0.7157	0.8621	0.0793	0.4736	0.6253	0.8254
Z_i	−1.165	−0.132	0.570	1.090	−1.410	−0.066	0.319	0.936

Observe that there is no U in the first quartile, 4 U's in the second quartile, 1 U in the third quartile, and 3 U's in the fourth quartile. Hence, the V's seem to be more uniform.

For Questions **58** and **59**, you are to assume the Black-Scholes framework.

Let $C(K)$ denote the Black-Scholes price for a 3-month K-strike European call option on a nondividend-paying stock.

Let $\hat{C}(K)$ denote the Monte Carlo price for a 3-month K-strike European call option on the stock, calculated by using 5 random 3-month stock prices simulated under the risk-neutral probability measure.

You are to estimate the price of a 3-month 42-strike European call option on the stock using the formula

$$C^*(42) = \hat{C}(42) + \beta[C(40) - \hat{C}(40)],$$

where the coefficient β is such that the variance of $C^*(42)$ is minimized.

You are given:
   (i)   The continuously compounded risk-free interest rate is 8%.
   (ii) $C(40) = 2.7847$.
   (iii)   Both Monte Carlo prices, $\hat{C}(40)$ and $\hat{C}(42)$, are calculated using the
          following 5 random 3-month stock prices:
                33.29,   37.30,   40.35,   43.65,   48.90

58.   Based on the 5 simulated stock prices, estimate β.

   (A) Less than 0.75

   (B) At least 0.75, but less than 0.8

   (C) At least 0.8, but less than 0.85

   (D) At least 0.85, but less than 0.9

   (E) At least 0.9

59.   Based on the 5 simulated stock prices, compute $C^*(42)$.

   (A) Less than 1.7

   (B) At least 1.7, but less than 1.9

   (C) At least 1.9, but less than 2.2

   (D) At least 2.2, but less than 2.6

   (E) At least 2.6

Solution to (58)   Answer: (B)

$$\text{Var}[C^*(42)] = \text{Var}[\hat{C}(42)] + \beta^2\text{Var}[\hat{C}(40)] - 2\beta\text{Cov}[\hat{C}(42),\hat{C}(40)],$$

which is a quadratic polynomial of β. (See also (19.11) in McDonald.) The minimum of the polynomial is attained at

$$\beta = \text{Cov}[\hat{C}(40), \hat{C}(42)]/\text{Var}[\hat{C}(40)].$$

For a pair of random variables X and Y, we estimate the ratio, $\text{Cov}[X, Y]/\text{Var}[X]$, using the formula

$$\frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sum_{i=1}^{n}(X_i-\bar{X})^2} = \frac{\sum_{i=1}^{n}X_iY_i - n\bar{X}\bar{Y}}{\sum_{i=1}^{n}X_i^2 - n\bar{X}^2}.$$

We now treat the payoff of the 40-strike option (whose correct price, $C(40)$, is known) as X, and the payoff of the 42-strike option as Y. We do not need to discount the payoffs because the effect of discounting is canceled in the formula above.

Simulated $S(0.25)$	$\max(S(0.25) - 40, 0)$	$\max(S(0.25) - 42, 0)$
33.29	0	0
37.30	0	0
40.35	0.35	0
43.65	3.65	1.65
48.90	8.9	6.9

We have $\bar{X} = \dfrac{0.35+3.65+8.9}{5} = 2.58$, $\bar{Y} = \dfrac{1.65+6.9}{5} = 1.71$,

$\sum_{i=1}^{n}X_i^2 = 0.35^2 + 3.65^2 + 8.9^2 = 92.655$, and $\sum_{i=1}^{n}X_iY_i = 3.65\times1.65 + 8.9\times6.9 = 67.4325$.

So, the estimate for the minimum-variance coefficient β is

$$\frac{67.4325 - 5\times2.58\times1.71}{92.655 - 5\times2.58^2} = 0.764211.$$

Remark: The estimate for the minimum-variance coefficient β can be obtained by using the statistics mode of a scientific calculator very easily. In the following we use TI–30X IIB as an illustration.

Step 1: Press [2nd][DATA] and select "2-VAR".

Step 2: Enter the five data points by the following keystroke:

[ENTER][DATA] 0 ↓ 0 ↓ 0 ↓ 0 ↓ 0.35 ↓ 0 ↓ 3.65 ↓ 1.65 ↓ 8.9 ↓ 6.9 [Enter]

Step 3: Press [STATVAR] and look for the value of "a".

Step 4: Press [2nd][STATVAR] and select "Y" to exit the statistics mode.
You can also find $\overline{X}$, $\overline{Y}$, $\sum_{i=1}^{n} X_i Y_i$, $\sum_{i=1}^{n} X_i^2$ etc in [STATVAR] too.

Below are keystrokes for TI–30XS multiview

Step 1: Enter the five data points by the following keystrokes:

[DATA] 0 ↓ 0 ↓ 0.35 ↓ 3.65 ↓ 8.9 ↓ → 0 ↓ 0 ↓ 0 ↓ 1.65 ↓ 6.9 [Enter]
Step 2: Press [2nd][STAT] and select "2-VAR".

Step 3: Select L1 and L2 for *x* and *y* data. Then select Calc and [ENTER]

Step 4: Look for the value of "a" by scrolling down.

Solution to (59)   Answer: (B)

The plain-vanilla Monte Carlo estimates of the two call option prices are:

$$\text{For } K = 40: e^{-0.08 \times 0.25} \times \frac{0.35 + 3.65 + 8.9}{5} = 2.528913$$

$$\text{For } K = 42: e^{-0.08 \times 0.25} \times \frac{1.65 + 6.9}{5} = 1.676140$$

The minimum-variance control variate estimate is
$$C^*(42) = \hat{C}(42) + \beta[C(40) - \hat{C}(40)]$$
$$= 1.6761 + 0.764211 \times (2.7847 - 2.5289)$$
$$= 1.872.$$

60. The short-rate process $\{r(t)\}$ in a Cox-Ingersoll-Ross model follows

$$dr(t) = [0.011 - 0.1r(t)]dt + 0.08\sqrt{r(t)}\,dZ(t),$$

where $\{Z(t)\}$ is a standard Brownian motion under the true probability measure.

For $t \le T$, let $P(r,t,T)$ denote the price at time t of a zero-coupon bond that pays 1 at time T, if the short-rate at time t is r.

You are given:

(i)   The Sharpe ratio takes the form $\phi(r,t) = c\sqrt{r}$.

(ii) $\lim\limits_{T\to\infty} \dfrac{1}{T}\ln[P(r,0,T)] = -0.1$ for each $r > 0$.

Find the constant c.

(A)   0.02

(B)   0.07

(C)   0.12

(D)   0.18

(E)   0.24

Solution to (60)   Answer: (E)

From the stochastic differential equation,
$$a(b - r) = 0.011 - 0.1r;$$
hence,
$$a = 0.1 \text{ and } b = 0.11.$$
Also, $\sigma = 0.08$.

Let $y(r, 0, T)$ be the continuously compounded yield rate of $P(r, 0, T)$, i.e.,
$$e^{-y(r, 0, T)T} = P(r, 0, T).$$
Then condition (ii) is
$$\lim_{T \to \infty} y(r, 0, T) = \lim_{T \to \infty} \frac{-\ln P(r, 0, T)}{T} = 0.1.$$

According to lines 10 to 12 on page 788 of McDonald (2006),
$$\lim_{T \to \infty} y(r, 0, T) = \frac{2ab}{a - \bar{\phi} + \gamma}$$
$$= \frac{2ab}{a - \bar{\phi} + \sqrt{(a - \bar{\phi})^2 + 2\sigma^2}},$$
where $\bar{\phi}$ is a positive constant such that the Sharpe ratio takes the form
$$\phi(r, t) = \bar{\phi}\sqrt{r} / \sigma.$$

Hence,
$$0.1 = \frac{2(0.1)(0.11)}{0.1 - \bar{\phi} + \sqrt{(0.1 - \bar{\phi})^2 + 2(0.08)^2}}.$$
We now solve for $\bar{\phi}$:
$$(0.1 - \bar{\phi}) + \sqrt{(0.1 - \bar{\phi})^2 + 2(0.08)^2} = 0.22$$
$$(0.1 - \bar{\phi})^2 - 0.44(0.1 - \bar{\phi}) + 0.0484 = (0.1 - \bar{\phi})^2 + 0.0128$$
$$0.44(0.1 - \bar{\phi}) = 0.0356$$
$$\bar{\phi} = 0.01909$$
Condition (i) is
$$\phi(r, t) = c\sqrt{r}.$$
Thus,
$$c = \bar{\phi} / \sigma$$
$$= 0.01909 / 0.08$$
$$= 0.2386.$$

Remarks: (i) The answer can be obtained by trial-and-error. There is no need to solve the quadratic equation.

(ii) If your textbook is an earlier printing of the second edition, you will find the corrected formulas in

http://www.kellogg.northwestern.edu/faculty/mcdonald/htm/p780-88.pdf

(iii) Let

$$y(r,t,T) = -\frac{1}{T-t}\ln P(r,t,T).$$

We shall show that

$$\lim_{T\to\infty} y(r,t,T) = \frac{2ab}{a-\bar\phi+\sqrt{(a-\bar\phi)^2+2\sigma^2}}.$$

Under the CIR model, the zero-coupon bond price is of the "affine" form

$$P(r,\,t,\,T) = A(t,\,T)e^{-B(t,\,T)r}.$$

Hence,

$$y(r,t,T) = -\frac{1}{T-t}\ln A(t,T) + \frac{r}{T-t}B(t,T). \qquad (1)$$

Observe that

$$\frac{1}{T-t}\ln A(t,T) = \frac{2ab}{\sigma^2(T-t)}\ln\left[\frac{2\gamma e^{(a-\bar\phi+\gamma)(T-t)/2}}{(a-\bar\phi+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}\right]$$

$$= \frac{2ab}{\sigma^2}\left\{\frac{\ln 2\gamma}{T-t}+\frac{a-\bar\phi+\gamma}{2}-\frac{\ln[(a-\bar\phi+\gamma)(e^{\gamma(T-t)}-1)+2\gamma]}{T-t}\right\}$$

where $\gamma > 0$. By applying l'Hôpital's rule to the last term above, we get

$$\lim_{T\to\infty}\frac{\ln[(a-\bar\phi+\gamma)(e^{\gamma(T-t)}-1)+2\gamma]}{T-t} = \lim_{T\to\infty}\frac{\gamma(a-\bar\phi+\gamma)e^{\gamma(T-t)}}{(a-\bar\phi+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}$$

$$= \lim_{T\to\infty}\frac{\gamma(a-\bar\phi+\gamma)}{(a-\bar\phi+\gamma)(1-e^{-\gamma(T-t)})+2\gamma e^{-\gamma(T-t)}}$$

$$= \frac{\gamma(a-\bar\phi+\gamma)}{(a-\bar\phi+\gamma)(1-0)+2\gamma\cdot 0}$$

$$= \gamma.$$

So,

$$\lim_{T\to\infty}\frac{1}{T-t}\ln A(t,T) = \frac{2ab}{\sigma^2}\left[0+\frac{a-\bar\phi+\gamma}{2}-\gamma\right] = \frac{ab(a-\bar\phi-\gamma)}{\sigma^2}.$$

We now consider the last term in (1). Since

$$B(t,T) = \frac{2(e^{\gamma(T-t)} - 1)}{(a - \bar{\phi} + \gamma)(e^{\gamma(T-t)} - 1) + 2\gamma} = \frac{2(1 - e^{-\gamma(T-t)})}{(a - \bar{\phi} + \gamma)(1 - e^{-\gamma(T-t)}) + 2\gamma e^{-\gamma(T-t)}},$$

we have

$$\lim_{T \to \infty} B(t,T) = \frac{2}{a - \bar{\phi} + \gamma}$$

and the limit of the second term in (1) is 0. Gathering all the results above,

$$\lim_{T \to \infty} y(r,t,T) = -\frac{ab(a - \bar{\phi} - \gamma)}{\sigma^2},$$

where

$$\gamma = \sqrt{(a - \bar{\phi})^2 + 2\sigma^2}.$$

To obtain the expression in McDonald (2006), consider

$$\frac{ab(a - \bar{\phi} - \gamma)}{\sigma^2} \cdot \frac{a - \bar{\phi} + \gamma}{a - \bar{\phi} + \gamma} = \frac{ab[(a - \bar{\phi})^2 - \gamma^2]}{\sigma^2(a - \bar{\phi} + \gamma)} = \frac{ab(-2)}{a - \bar{\phi} + \gamma}.$$

Thus we have

$$\lim_{T \to \infty} y(r,t,T) = \frac{2ab}{a - \bar{\phi} + \gamma}$$

$$= \frac{2ab}{a - \bar{\phi} + \sqrt{(a - \bar{\phi})^2 + 2\sigma^2}}.$$

Note that this "long" term interest rate does not depend on r or on t.

61. Assume the Black-Scholes framework.

You are given:

(i) $S(t)$ is the price of a stock at time t.

(ii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 1%.

(iii) The stock-price process is given by

$$\frac{dS(t)}{S(t)} = 0.05dt + 0.25dZ(t)$$

where $\{Z(t)\}$ is a standard Brownian motion under the true probability measure.

(iv) Under the risk-neutral probability measure, the mean of $Z(0.5)$ is -0.03.

Calculate the continuously compounded risk-free interest rate.

(A)   0.030

(B)   0.035

(C)   0.040

(D)   0.045

(E)   0.050

Solution to (61)   Answer: (D)

Let α, δ, and σ be the stock's expected rate of (total) return, dividend yield, and volatility, respectively.
From (ii), $\delta = 0.01$.
From (iii), $\alpha - \delta = 0.05$; hence, $\alpha = 0.06$.
Also from (iii), $\sigma = 0.25$.

Thus, the Sharpe ratio is

$$\eta = \frac{\alpha - r}{\sigma} = \frac{0.06 - r}{0.25} = 0.24 - 4r. \tag{1}$$

According to Section 20.5 in McDonald (2006), the stochastic process $\{\tilde{Z}(t)\}$ defined by

$$\tilde{Z}(t) = Z(t) + \eta t \tag{2}$$

is a standard Brownian motion under the risk-neutral probability measure; see, in particular, the third paragraph on page 662. Thus,

$$E^*[\tilde{Z}(t)] = 0, \tag{3}$$

where the asterisk signifies that the expectation is taken with respect to the risk-neutral probability measure.

The left-hand side of equation (3) is

$$E^*[Z(t)] + \eta t = E^*[Z(t)] + (0.24 - 4r)t \tag{4}$$

by (1). With $t = 0.5$ and applying condition (iii), we obtain from (3) and (4) that

$$-0.03 + (0.24 - 4r)(0.5) = 0,$$

yielding $r = 0.045$.

Remark In Section 24.1 of McDonald (2006), the Sharpe ratio is not a constant but depends on time t and the short-rate. Equation (2) becomes

$$\tilde{Z}(t) = Z(t) - \int_0^t \phi[r(s), s]ds. \tag{5}$$

$\{Z(t)\}$ is a standard Brownian motion under the true probability measure, and $\{\tilde{Z}(t)\}$ is a standard Brownian motion under the risk-neutral probability measure. Note that in (5) there is a minus sign, instead of a plus sign as in (2); this is due to the minus sign in (24.1).

62. Assume the Black-Scholes framework.

Let $S(t)$ be the time-t price of a stock that pays dividends continuously at a rate proportional to its price.

You are given:

(i) $\qquad \dfrac{dS(t)}{S(t)} = \mu dt + 0.4 d\tilde{Z}(t),$

where $\{\tilde{Z}(t)\}$ is a standard Brownian motion under the risk-neutral probability measure.

(ii) For $0 \le t \le T$, the time-t forward price for a forward contract that delivers the square of the stock price at time T is

$$F_{t,T}(S^2) = S^2(t)\exp[0.18(T - t)].$$

Calculate the constant μ.

(A) 0.01

(B) 0.04

(C) 0.07

(D) 0.10

(E) 0.40

Solution to (62)   Answer: (A)

By comparing the stochastic differential equation in (i) with equation (20.26) in McDonald (2006), we have

$$\mu = r - \delta$$

and

$$\sigma = 0.4.$$

The time-*t* prepaid forward price for the forward contract that delivers S^2 at time *T* is

$$F_{t,T}^P(S^2) = e^{-r(T-t)}F_{t,T}(S^2) = S^2(t)\exp[(0.18 - r)(T - t)].$$

For a derivative security that does not pay dividends, the prepaid forward price is the price. Thus, the prepaid forward price satisfies the Black-Scholes partial differential equation (21.11),

$$\frac{\partial V}{\partial t} + (r - \delta)s\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} = rV.$$

The partial derivatives of $V(s, t) = s^2\exp[(0.18 - r)(T - t)]$, $t \le T$, for the partial differential equation are:

$$V_t = (r - 0.18)V(s, t),$$

$$V_s = 2s \times \exp[(0.18 - r)(T - t)] = \frac{2V(s,t)}{s},$$

$$V_{ss} = 2\exp[(0.18 - r)(T - t)] = \frac{2V(s,t)}{s^2}.$$

Substituting these derivatives into the partial differential equation yields

$$(r - 0.18)V(s,t) + (r - \delta)s\frac{2V(s,t)}{s} + \frac{1}{2}\sigma^2 s^2 \frac{2V(s,t)}{s^2} = rV(s,t)$$

$$\Rightarrow (r - 0.18) + 2(r - \delta) + \sigma^2 = r$$

$$\Rightarrow r - \delta = \frac{0.18 - \sigma^2}{2} = 0.01.$$

Alternative Solution: Compare the formula in condition (ii) (with $t = 0$) with McDonald's formula (20.31) (with $a = 2$) to obtain the equation

$$0.18 = 2(r - \delta) + \frac{1}{2} \times 2(2 - 1)\sigma^2,$$

which again yields

$$r - \delta = \frac{0.18 - \sigma^2}{2}.$$

Remarks:

(i) An easy way to obtain (20.31) is to use the fact that, because the risk-free interest rate is constant,

$$F_{0,T}(S^a) = E^*[[S(T)]^a],$$

where the asterisk signifies that the expectation is taken with respect to the risk-neutral probability measure. Under the risk-neutral probability measure, $\ln[S(T)/S(0)]$ is a normal random variable with mean $(r - \delta - \frac{1}{2}\sigma^2)T$ and variance $\sigma^2 T$. Thus, by (18.13) or by the moment-generating function formula for a normal random variable, we have

$$F_{0,T}(S^a) = E^*[[S(T)]^a] = [S(0)]^2 \exp[a(r - \delta - \frac{1}{2}\sigma^2)T + \frac{1}{2}\times a^2\sigma^2 T],$$

which is (20.31).

(ii) We can also use the partial differential equation (21.34) to solve the problem. Any constant times $\exp(-rt)$ times any solution of (21.11) gives a solution for (21.34). Thus, we can use

$$V(t, s) = s^2\exp[0.18(T - t)],$$

which is the given formula for the forward price (or futures price). Because T is a constant, we use

$$V(t, s) = s^2 e^{-0.18t};$$

then (21.34) becomes

$$-0.18V + \sigma^2 V + (r - \delta)2V = 0.$$

Equation (21.34) is not in the current syllabus of Exam MFE/3F.

(iii) Another way to determine $r - \delta$ is to use the fact that, for a security that does not pay dividends, its discounted price process is a martingale under the risk-neutral probability measure. Thus, the stochastic process $\{e^{-rt}F_{t,T}^P(S^2); 0 \le t \le T\}$ is a martingale. Because

$$e^{-rt}F_{t,T}^P(S^2) = e^{-rT}F_{t,T}(S^2)$$

$$= e^{-rT}[S(t)]^2 \exp[0.18(T - t)]$$

$$= e^{(0.18-r)T}[S(t)]^2 e^{-0.18t},$$

the process $\{[S(t)]^2 e^{-0.18t}; 0 \le t \le T\}$ is also a martingale. The martingale condition is

$$[S(0)]^2 e^0 = E^*[[S(t)]^2 e^{-0.18t}]$$

$$= e^{-0.18t}E^*[[S(t)]^2]$$

$$= e^{-0.18t}[S(0)]^2 \exp[2(r - \delta - \frac{1}{2}\sigma^2)t + \frac{1}{2}\times 2^2\sigma^2 t],$$

or

$$0 = -0.18t + 2(r - \delta - \frac{1}{2}\sigma^2)t + \frac{1}{2}\times 2^2\sigma^2 t,$$

which again leads to

$$r - \delta = \frac{0.18 - \sigma^2}{2}.$$

This method is beyond the current syllabus of Exam MFE/3F.

63. Define

    (i) $W(t) = t^2$.

    (ii) $X(t) = [t]$, where $[t]$ is the greatest integer part of t; for example, $[3.14] = 3$, $[9.99] = 9$, and $[4] = 4$.

    (iii) $Y(t) = 2t + 0.9Z(t)$, where $\{Z(t): t \geq 0\}$ is a standard Brownian motion.

Let $V_T^2(U)$ denote the quadratic variation of a process U over the time interval $[0, T]$.

Rank the quadratic variations of W, X and Y over the time interval $[0, 2.4]$.

    (A) $V_{2.4}^2(W) < V_{2.4}^2(Y) < V_{2.4}^2(X)$

    (B) $V_{2.4}^2(W) < V_{2.4}^2(X) < V_{2.4}^2(Y)$

    (C) $V_{2.4}^2(X) < V_{2.4}^2(W) < V_{2.4}^2(Y)$

    (D) $V_{2.4}^2(X) < V_{2.4}^2(Y) < V_{2.4}^2(W)$

    (E)   None of the above.

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Solution to (63)   Answer: (A)

For a process $\{U(t)\}$, the quadratic variation over $[0, T]$, $T > 0$, can be calculated as

$$\int_0^T [dU(t)]^2.$$

(i)   Since
$$dW(t) = 2tdt,$$
we have
$$[dW(t)]^2 = 4t^2(dt)^2$$
which is zero, because $dt \times dt = 0$ by (20.17b) on page 658 of McDonald (2006).

This means $V_{2.4}^2(W) = \int_0^{2.4} [dW(t)]^2 = 0$.

(ii) $X(t) = 0$ for $0 \leq t < 1$, $X(t) = 1$ for $1 \leq t < 2$, $X(t) = 2$ for $2 \leq t < 3$, etc.
For $t \geq 0$, $dX(t) = 0$ except for the points 1, 2, 3, … . At the points 1, 2, 3, … , the square of the increment is $1^2 = 1$. Thus,
$$V_{2.4}^2(X) = 1 + 1 = 2.$$

(iii) By Itô's lemma,
$$dY(t) = 2dt + 0.9dZ(t).$$
By (20.17a, b, c) in McDonald (2006),
$$[dY(t)]^2 = 0.9^2 dt$$
Thus,
$$\int_0^{2.4} [dY(t)]^2 = \int_0^{2.4} 0.9^2 dt = 0.81 \times 2.4 = 1.944.$$

64. Let $S(t)$ denote the time-t price of a stock. Let $Y(t) = [S(t)]^2$. You are given

$$\frac{dY(t)}{Y(t)} = 1.2dt - 0.5dZ(t), \qquad Y(0) = 64,$$

where $\{Z(t): t \geq 0\}$ is a standard Brownian motion.

Let (L, U) be the 90% lognormal confidence interval for $S(2)$.

Find U.

(A) 27.97

(B) 33.38

(C) 41.93

(D) 46.87

(E) 53.35

Solution to (64)   Answer: (C)

For a given value of $V(0)$, the solution to the stochastic differential equation

$$\frac{dV(t)}{V(t)} = \mu\, dt + \beta\, dZ(t), \qquad t \geq 0, \tag{1}$$

is

$$V(t) = V(0)\exp[(\mu - \tfrac{1}{2}\beta^2)t + \beta Z(t)], \qquad t \geq 0. \tag{2}$$

That formula (2) satisfies equation (1) is a consequence of Itô's Lemma. See also Example 20.1 on p. 665 in McDonald (2006)

It follows from (2) that
$$Y(t) = 64\exp[(1.2 - \tfrac{1}{2}(-0.5)^2 t - 0.5Z(t)] = 64\exp[1.075t - 0.5Z(t)].$$

Since $Y(t) = [S(t)]^2$,
$$S(t) = 8\exp[0.5375t - 0.25Z(t)]. \tag{3}$$

Because $Z(2) \sim \text{Normal}(0, 2)$ and because $N^{-1}(0.95) = 1.645$, we have
$$\Pr(-1.645\sqrt{2} < Z(2) < 1.645\sqrt{2}) = 0.90.$$

Hence,
$$\Pr(L < S(2) < U) = 0.90,$$

if
$$U = 8 \times \exp(1.075 + 0.25 \times 1.645\sqrt{2}) = 41.9315$$

and
$$L = 8 \times \exp(1.075 - 0.25 \times 1.645\sqrt{2}) = 13.1031.$$

Remarks: (i) It is more correct to write the probability as a conditional probability,
$$\Pr(13.1031 < S(2) < 41.9315 \mid S(0) = 8) = 0.90.$$
(ii) The term "confidence interval" as used in Section 18.4 seems incorrect, because $S(2)$ is a random variable, not an unknown, but constant, parameter. The expression
$$\Pr(L < S(2) < U) = 0.90$$
gives the probability that the random variable $S(2)$ is between L and U, not the "confidence" for $S(2)$ to be between L and U.
(iii) By matching the right-hand side of (20.32) (with $a = 2$) with the right-hand side of the given stochastic differential equation, we have $\sigma = -0.25$ and $\alpha - \delta = 0.56875$. It then follows from (20.29) that
$$S(t) = 8 \times \exp[(0.56875 - \tfrac{1}{2}(-0.25)^2)t - 0.25Z(t)],$$
which is (3) above.

65.   Assume the Black-Scholes framework.

You are given:

(i) $S(t)$ is the time-t price of a stock, $t \geq 0$.

(ii)   The stock pays dividends continuously at a rate proportional to its price.

(iii)   Under the true probability measure, $\ln[S(2)/S(1)]$ is a normal random variable with mean 0.10.

(iv)   Under the risk-neutral probability measure, $\ln[S(5)/S(3)]$ is a normal random variable with mean 0.06.

(v)   The continuously compounded risk-free interest rate is 4%.

(vi)   The time-0 price of a European put option on the stock is 10.

(vii)   For delta-hedging at time 0 one unit of the put option with shares of the stock, the cost of stock shares is 20.

Calculate the absolute value of the time-0 continuously compounded expected rate of return on the put option.

(A)   4%

(B)   5%

(C)   10%

(D)   11%

(E)   18%

Solution to (65)   Answer: (C)

Equation (21.22) on page 687 of McDonald (2006) is

$$\alpha_{option} - r = \frac{SV_S}{V}(\alpha - r).$$

Statement (iii) means that $(\alpha - \delta - \tfrac{1}{2}\sigma^2) \times (2 - 1) = 0.10$.

Statement (iv) means that $(r - \delta - \tfrac{1}{2}\sigma^2) \times (5 - 3) = 0.06$.

Hence, $\alpha - r = 0.10 - (0.06/2) = 0.07$.

Statement (vii) means that $SV_S = -20$ (There is a minus sign because $V_S = -e^{-\delta T}N(-d_1)$, a negative quantity.) Thus,

$$\alpha_{option} = r + \frac{SV_S}{V}(\alpha - r) = 0.04 + \frac{-20}{10}(0.07) = -10\%.$$

Remark: In Chapter 12, equation (21.22) is written as (12.10). As stated on page 687, (21.22) states an *equilibrium* condition that the option price must obey.

66. Consider two stocks X and Y. There is a single source of uncertainty which is captured by a standard Brownian motion $\{Z(t)\}$.

You are given:

(i) Each stock pays dividends continuously at a rate proportional to its price. For X, the dividend yield is 2%. For Y, the dividend yield is 1%.

(ii) The time-t price of X satisfies

$$\frac{dX(t)}{X(t)} = 0.06dt + 0.2dZ(t).$$

(iii) The time-t price of Y is

$$Y(t) = Y(0)e^{\mu t - 0.1Z(t)}.$$

(iv) The continuously compounded risk-free interest rate is 4%.

Calculate the constant μ.

(A)   0.005

(B)   0.010

(C)   0.015

(D)   0.025

(E)   0.045

Solution to (66)   Answer: (A)

Differentiating the equation in (iii) using Itô's Lemma (or from Example 20.1 on page 665 of McDonald (2006)), we have

$$\frac{dY(t)}{Y(t)} = [\mu + \frac{1}{2}(-0.1)^2]dt - 0.1dZ(t)$$

$$= (\mu + 0.005)dt - 0.1dZ(t).$$

Because Y's dividend yield is 0.01, we rewrite the last equation as

$$\frac{dY(t)}{Y(t)} = [(\mu + 0.015) - 0.01]dt - 0.1dZ(t),$$

so that it is comparable to (20.25). Similarly, we rewrite the equation in (ii) as

$$\frac{dX(t)}{X(t)} = (0.08 - 0.02)dt + 0.2dZ(t).$$

The no-arbitrage argument in Section 20.4 "The Sharpe Ratio" of McDonald (2006) shows that

$$\frac{\mu + 0.015 - r}{-0.1} = \frac{0.08 - r}{0.2}.$$

By (iv), $r = 0.04$. Thus, the equation becomes

$$\frac{\mu + 0.015 - 0.04}{-0.1} = \frac{0.08 - 0.04}{0.2},$$

or

$$\mu = 0.005.$$

67. Assume the Black-Scholes framework.

- In a securities market model, there are two stocks, S_1 and S_2, whose price processes are:

$$\frac{dS_1(t)}{S_1(t)} = \mu[dt + 20dZ(t)],$$

$$d[\ln S_2(t)] = 0.03dt + 0.2dZ(t),$$

  where μ is a constant and $\{Z(t)\}$ is a standard Brownian motion.

- S_1 does not pay dividends.

- For $t \geq 0$, S_2 pays dividends of amount $0.01S_2(t)dt$ between time t and time $t + dt$.

- The continuously compounded risk-free interest rate is 4%.

Determine μ.

(A) −0.04

(B) −0.02

(C) 0

(D) 0.02

(E) 0.04

Solution to (67)   Answer: (A)

Define $Y(t) = \ln S_2(t)$. Differentiating $S_2(t) = e^{Y(t)}$ by means of Itô's lemma, we have

$$dS_2(t) = e^{Y(t)}dY(t) + \frac{1}{2}e^{Y(t)}[dY(t)]^2 + 0dt$$

$$= S_2(t)[0.03dt + 0.2dZ(t)] + \frac{1}{2}S_2(t)[0.2^2dt]$$

$$= 0.05S_2(t)dt + 0.2S_2(t)dZ(t).$$

Hence, $\alpha_2 - \delta_2 = 0.05$. Since it is given that $\delta_2 = 0.01$, we obtain
$$\alpha_2 = \delta_2 + 0.05 = 0.06.$$

The no-arbitrage argument in Section 20.4 "The Sharpe Ratio" of McDonald (2006) shows that

$$\frac{\mu - r}{20\mu} = \frac{0.06 - r}{0.2},$$

With $r = 0.04$, we obtain
$$\mu = -0.04.$$

Remarks: (i) An alternative solution: Integrating
$$d[\ln S_2(\tau)] = 0.03d\tau + 0.2dZ(\tau)$$
from $\tau = 0$ to $\tau = t$ yields
$$\ln S_2(t) - \ln S_2(0) = 0.03(t - 0) + 0.2[Z(t) - Z(0)].$$
Because $Z(0) = 0$ by definition, we obtain
$$S_2(t) = S_2(0)\exp[0.03t + 0.2Z(t)].$$
It follows from (20.29) of McDonald (2006) or from Itô's lemma that
$$0.03 = \alpha_2 - \delta_2 - \frac{1}{2}\sigma_2^2 = \alpha_2 - \delta_2 - \frac{1}{2}(0.2)^2,$$
which leads to $\alpha_2 = \delta_2 + 0.05$ again.

(ii) In the last paragraph on page 394 of McDonald (2006), the Sharpe ratio for an asset is defined as "the ratio of the risk premium to volatility." (On page 917, the term "return standard deviation" is used in place of "volatility.") If the price process of an asset S satisfies

$$\frac{dS(t)}{S(t)} = a(t)dt + b(t)dZ(t)$$

and if S pays dividends of amount $\delta(t)S(t)dt$ between time t and $t + dt$, then its time-t Sharpe ratio is

$$\frac{a(t) + \delta(t) - r(t)}{|b(t)|},$$

where $r(t)$ is the short-rate at time t. In Section 24.1, this formula takes the form

$$\frac{\alpha(r(t), t, T) - r(t)}{q(r(t), t, T)}.$$

In this sample problem, the Sharpe ratio for S_1 is

$$\frac{\mu - r}{|20\mu|} = \frac{-0.04 - 0.04}{20 \times 0.04} = -0.1,$$

and the Sharpe ratio for S_2 is

$$\frac{0.06 - 0.04}{0.2} = 0.1,$$

which is the negative of that for S_2.

(iii) For a derivative security or an option on a stock, the Sharpe ratio is

$$\frac{\alpha_{\text{option}} - r}{\sigma_{\text{option}}} = \frac{\Omega \times (\alpha_{\text{stock}} - r)}{|\Omega| \times \sigma_{\text{stock}}} = \text{sign}(\Omega) \times \frac{\alpha_{\text{stock}} - r}{\sigma_{\text{stock}}},$$

where $\Omega = SV_S/V$. The formula above is a corrected version of (21.21) on page 687; see also (12.12) on page 394. This shows that some derivative securities, such as put options, have a Sharpe ratio that is the negative of that for the stock, while others, such as call options, have the same Sharpe ratio as the stock.

(iv) Some authors call the ratio

$$\frac{a(t) + \delta(t) - r(t)}{b(t)},$$

where there is no absolute value in the denominator, the *market price of risk* or *market price for risk*. The argument in Section 20.4 shows that the condition of no arbitrage implies that, at each t, all tradable assets have the same market price of risk. Three examples of this important result are:

$$\frac{\mu - r}{20\mu} = \frac{0.06 - r}{0.2},$$

which is the equation we used to solve the problem;

$$\frac{\alpha(r(t), t, T_1) - r(t)}{-q(r(t), t, T_1)} = \frac{\alpha(r(t), t, T_2) - r(t)}{-q(r(t), t, T_2)},$$

which is equivalent to equation (24.16) in McDonald (2006); and

$$\frac{\alpha_{\text{stock}} - r}{\sigma_{\text{stock}}} = \frac{\alpha_{\text{option}} - r}{\text{sign}(\Omega) \times \sigma_{\text{option}}},$$

which is a corrected version of (21.21). The term "market price of risk" is not in the current syllabus of MFE/3F.

68. Consider the stochastic differential equation

$$dX(t) = -3X(t)dt + 2dZ(t),$$

where $\{Z(t)\}$ is a standard Brownian motion.

You are given that a solution is

$$X(t) = e^{-At}\left[B + C\int_0^t e^{Ds}\,dZ(s)\right],$$

where A, B, C and D are constants.

Calculate the sum $A + C + D$.

(A) 2

(B) 4

(C) 7

(D) 8

(E) 11

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Solution to (68)   Answer: (D)

With the definitions

$$Y(t) = \int_0^t e^{Ds} dZ(s)$$

and

$$g(y, t) = e^{-At}(B + Cy),$$

the given solution can be expressed as

$$X(t) = g(Y(t), t),$$

which we now differentiate by means of Itô's lemma. Because

$$g_t(y, t) = -Ae^{-At}(B + Cy) = -Ag(y, t),$$

$$g_y(y, t) = Ce^{-At}y,$$

$$g_{yy}(y, t) = 0,$$

and

$$dY(t) = e^{Dt} dZ(t),$$

we obtain

$$dg(Y(t), t) = g_y(Y(t), t)dY(t) + \tfrac{1}{2}g_{yy}(Y(t), t)[dY(t)]^2 + g_t(Y(t), t)dt$$

$$= Ce^{-At}e^{Dt}dZ(t) - Ag(Y(t), t)dt.$$

Replacing $g(Y(t), t)$ by $X(t)$, we have

$$dX(t) = Ce^{(D - A)t}dZ(t) - AX(t)dt,$$

comparing which with the given stochastic differential equation yields

$$A = 3, D = A = 3, \text{ and } C = 2.$$

Hence,

$$A + C + D = 8.$$

Remarks:

(i)   This question is related to Exercise 20.9 in McDonald (2006) and #24 in this set of sample questions.

(ii) $B = X(0)$.

(iii) $\{X(t)\}$ is an Ornstein-Uhlenbeck process.

69. Consider the following three-period binomial tree for modeling the price movements of a nondividend-paying stock. The length of each period is 1 year.

The continuously compounded risk-free interest rate is 10%.

Consider a 3-year at-the-money American put option on the stock.

Approximate its time-0 theta using the method in Appendix 13.B of McDonald (2006).
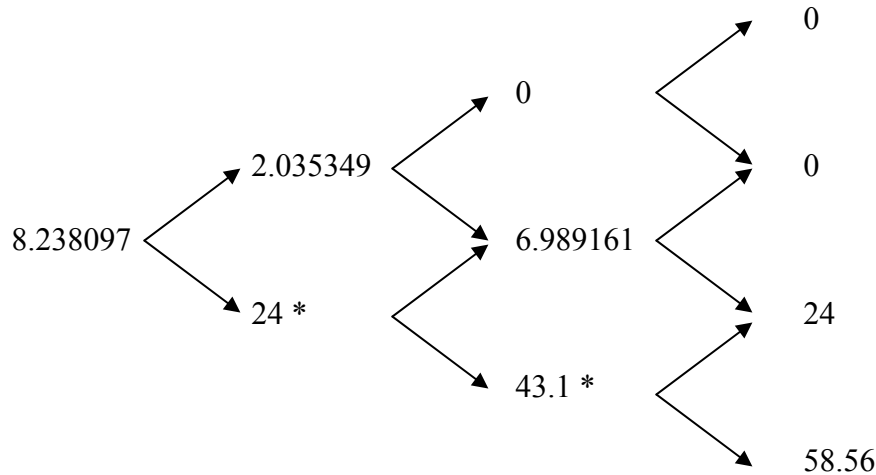
(A) −8.94

(B) −2.33

(C) −0.62

(D) 0.90

(E) 8.24

Solution to (69)   Answer: (C)

American options are priced by the method of backward induction. See Section 10.4 of McDonald (2006).

By (10.5), we have $p* = \dfrac{120e^{0.1} - 96}{150 - 96} = 0.678158$.

Applying (10.12) repeatedly, we obtain the binomial tree for pricing the American put option. (* means that early exercise is optimal.)

It is obvious that $P_{uu} = 0$. The other values are:

$P_{ud} = e^{-0.1}[24(1 - p*)] = 6.9891612$

$P_{dd} = \max\{e^{-0.1}[24p* + 58.56(1 - p*)],\ 43.1\} = \max\{31.7804,\ 43.1\} = 43.1$

$P_u = e^{-0.1}[6.9891612(1 - p*)] = 2.0353489$

$P_d = \max\{e^{-0.1}[6.9891612p* + 43.1(1 - p*)],\ 24\} = \max\{16.84,\ 24\} = 24$

$P_0 = e^{-0.1}[2.0353489p* + 24(1 - p*)] = 8.23809684$

Formula (13.17) approximates theta as

$$\frac{P(S_{ud},\ 2h) - [P(S,\ 0) + \varepsilon\Delta_P(S,0) + \tfrac{1}{2}\varepsilon^2\Gamma_P(S,0)]}{2h} .$$

With $h = 1$ and $\varepsilon = S_{ud} - S = 0$, the expression above becomes

$$\frac{P(S_{ud},\ 2h) - P(S,\ 0)}{2h} = \frac{P_{ud} - P_0}{2} = \frac{6.989161 - 8.238097}{2} = -0.624468 .$$

70. Assume the Black-Scholes framework.

You are given:

(i) $S(t)$ is the time-t price of a stock.

(ii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 2%.

(iii) $S(t)$ satisfies

$$\frac{dS(t)}{S(t)} = 0.1dt + 0.2dZ(t),$$

where $\{Z(t)\}$ is a standard Brownian motion.

(iv) An investor employs a proportional investment strategy. At every point of time, 80% of her assets are invested in the stock and 20% in a risk-free asset earning the risk-free rate.

(v) The continuously compounded risk-free interest rate is 5%.

Let $W(t)$ be the value of the investor's assets at time t, $t \geq 0$.

Determine $W(t)$.

(A) $W(0)e^{0.0772t+0.16Z(t)}$

(B) $W(0)e^{0.0900t+0.16Z(t)}$

(C) $W(0)e^{0.0932t+0.16Z(t)}$

(D) $W(0)e^{0.1060t+0.16Z(t)}$

(E) $W(0)e^{0.1200t+0.16Z(t)}$

Solution to (70)   Answer: (C)

For $t \geq 0$, the rate of return over the time interval from t to $t + dt$ is

$$\frac{dW(t)}{W(t)} = 0.8\left[\frac{dS(t)}{S(t)} + \delta dt\right] + 0.2rdt$$

$$= 0.8[0.12dt + 0.2dZ(t)] + 0.2(0.05)dt,$$

$$= 0.106dt + 0.16dZ(t)$$

By Example 20.1, the solution for $W(t)$ is

$$W(t) = W(0)\exp\left\{[0.106 - \frac{1}{2}(0.16)^2]t + 0.16Z(t)\right\} = W(0)e^{0.0932t+0.16Z(t)}.$$

Remark: Further discussion can be found in #32 of this sample set.

71. Assume the Black-Scholes framework. You are given:

(i)   For $t \geq 0$, $S(t)$ denotes the time-t price of a stock.

(ii) $S(0) = 1$

(iii) $\dfrac{dS(t)}{S(t)} = 0.08dt + 0.40d\tilde{Z}(t), \qquad t \geq 0,$

where $\{\tilde{Z}(t)\}$ is a standard Brownian motion under the risk-neutral probability measure.

(iv) For $t \geq 0$, the stock pays dividends of amount $0.04S(t)dt$ between time t and time $t + dt$.

(v)   For a real number c and a standard normal random variable Z,

$$E[Z^2 e^{cZ}] = (1 + c^2)e^{c^2/2}.$$

Consider a derivative security that pays
$$1 + S(1)\{\ln[S(1)]\}^2$$
at time $t = 1$, and nothing at any other time.

Calculate the time-0 price of this derivative security.

(A)   0.932

(B)   1.050

(C)   1.065

(D)   1.782

(E)   2.001

Solution to (71) Answer: (C)

The simplest way to price the security is to use the method of *risk neutral pricing*. That is, we compute the expectation of the discounted value of the payoff, where discounting is done with respect to the risk-free interest rate and expectation is taken with respect to the risk-neutral probability distribution. (See Section 21.3 of McDonald (2006), in particular, the first sentence in the second paragraph on page 692.)

Thus, the time-0 price is

$$E*[e^{-rT}(1 + S(T)\{\ln[S(T)]\}^2)],$$

where T is the exercise date (which is 1 in this problem) and

$$S(T) = S(0)\exp[(r - \delta - \frac{\sigma^2}{2})T + \sigma\tilde{Z}(T)].$$

Now, statement (iv) means $\delta = 0.04$.

From (21.28) and the stochastic differential equation given in (iii), we have $r - \delta = 0.08$, and $\sigma = 0.40$. Hence, we also have $r = 0.08 + \delta = 0.12$.

With $S(0) = 1$, we obtain

$$S(1) = e^{0.4\tilde{Z}(1)},$$

which means

$$1 + S(1)\{\ln[S(1)]\}^2 = 1 + e^{0.4\tilde{Z}(1)}(0.4\tilde{Z}(1))^2 = 1 + 0.16e^{0.4\tilde{Z}(1)}(\tilde{Z}(1))^2.$$

Thus, the time-0 price of the derivative security is

$$E*[e^{-0.12}[1 + 0.16e^{0.4\tilde{Z}(1)}(\tilde{Z}(1))^2]] = e^{-0.12}[1 + 0.16E(e^{0.4Z}Z^2)]$$

where Z is a standard normal random variable. By the formula given in (v) with $c = 0.4$,

$$\text{time-0 price} = e^{-0.12}[1 + 0.16(1 + 0.4^2)\exp(0.4^2/2)] = 1.0652,$$

which means (C) is the answer.

Remarks: (i) The formula given in statement (v), $E[Z^2 e^{cZ}] = (1 + c^2)e^{c^2/2}$, can be derived as follows:

$$E[Z^2 e^{cZ}] = \int_{-\infty}^{\infty} z^2 e^{cz} f_Z(z)dz$$

$$= \int_{-\infty}^{\infty} z^2 e^{cz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}dz$$

$$= e^{c^2/2} \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-(z-c)^2/2}dz$$

$$= e^{c^2/2}E(X^2),$$

where X is a normal random variable with mean c and variance 1. Because
$$E[X^2] = \text{Var}(X) + (EX)^2 = 1 + c^2,$$
we obtain the formula in (v).

(ii) Alternatively, we start with the moment-generating function formula for Z,
$$E(e^{cZ}) = \exp(c^2/2).$$
Differentiating both sides with respect to c, we get $E(Ze^{cZ}) = \exp(c^2/2) \times c$.

Differentiating again, we obtain
$$E(Z^2 e^{cZ}) = \exp(c^2/2)) \times c^2 + \exp(c^2/2) = (1 + c^2)\exp(c^2/2).$$
The method of differentiation under the expectation sign may be familiar to some Life Contingencies students:
$$\frac{\partial}{\partial \delta}\, \overline{A}_x = \frac{\partial}{\partial \delta} E[e^{-\delta T(x)}] = E[\frac{\partial}{\partial \delta} e^{-\delta T(x)}] = E[e^{-\delta T(x)} \times (-T(x))] = -(\overline{IA})_x\,,$$
where δ is the force of interest; more generally,
$$\frac{\partial}{\partial \delta}\, A_x^{(m)} = \frac{\partial}{\partial \delta} E[e^{-\delta\lceil mT(x)\rceil/m}] = -E[\frac{\lceil mT(x)\rceil}{m} e^{-\delta\lceil mT(x)\rceil/m}] = -(I^{(m)}A)_x^{(m)}\,.$$

(iii) Arguably, the most important result in the entire MFE/3F syllabus is that securities can be priced by the method of risk neutral pricing. This may seem a surprise to actuarial students because they have learned to value insurance liabilities by calculating their expected present values, and risk neutral pricing also means calculating expected present values. However, financial pricing is different from actuarial valuation in that the expectation is taken with respect to a risk-neutral probability measure and also that the discounting is computed with respect to the risk-free interest rate. Some authors call the following result the *fundamental theorem of asset pricing*: in a frictionless market, the absence of arbitrage is "essentially equivalent" to the existence of a risk-neutral probability measure with respect to which the price of a payoff is the expected discounted value. The term "fundamental theorem of asset pricing" is not in the current syllabus of exam MFE/3F.

72.   Assume the Black-Scholes framework.

The continuously compounded risk-free interest rate is 7%.

With respect to a stock S, you are given:

(i)   The current stock price, $S(0)$, is 10.

(ii)   The stock pays dividends continuously at a rate proportional to its price. The stock's volatility is 10%.

(iii)   The price of a 6-month European gap *call* option on S^2, with a strike price of 95 and a payment trigger of 120, is 5.543.

(iv)   The price of a 6-month European gap *put* option on S^2, with a strike price of 95 and a payment trigger of 120, is −4.745.

(v)   The strike price and payment trigger refer to the value of S^2, rather than S.

Calculate the dividend yield.

(A)   2.0%

(B)   3.5%

(C)   4.0%

(D)   4.5%

(E)   5.5%

Solution to (72)   Answer: (A)

Let K_1 denote the strike price (which is 95 in this problem), K_2 denote the payment trigger (which is 120), and T denote the exercise date (which is ½). Let $I(.)$ the denote the indicator function.

The payoff of the gap call is $(S^a(T) - K_1) \times I(S^a(T) > K_2)$ with $a = 2$.
The payoff of the gap put is $(K_1 - S^a(T)) \times I(S^a(T) < K_2) = -(S^a(T) - K_1)I(S^a(T) < K_2)$.

Let α be a real number, A be an event and $\tilde{A}$ be its complement. Obviously,
$$\alpha I(A) + \alpha I(\tilde{A}) = \alpha .$$
It follows from this identity that
$$(S^a(T) - K_1) \times I(S^a(T) > K_2) - (K_1 - S^a(T)) \times I(S^a(T) < K_2) = S^a(T) - K_1$$
(the right-hand side which is independent of K_2). By considering the prices of these time-T payoffs, we have a generalization of *put-call parity*:
$$\text{price of gap call} - \text{price of gap put} = F^P_{0,T}(S^a - K_1) = F^P_{0,T}(S^a) - K_1 e^{-rT} .$$

To evaluate $F^P_{0,T}(S^a)$, we use the "Black-Scholes framework" assumption. It follows from (20.30) of McDonald (2006) that
$$F^P_{0,T}(S^2) = S^2(0)e^{[-r+2(r-\delta)+\sigma^2]T} .$$

Solving the equation
$$5.543 - (-4.745) = (10)^2 e^{[-0.07+2(0.07-\delta)+0.01] \times \frac{1}{2}} - 95e^{-0.07 \times \frac{1}{2}}$$
yields
$$\delta = 0.02 .$$

Remark: Formula (20.30) of McDonald (2006) can be derived as follows:
$$\begin{aligned} F^P_{0,T}(S^a) &= E^*[e^{-rT} S^a(T)] \\ &= e^{-rT} S^a(0) E^*[e^{a[(r-\delta-\frac{1}{2}\sigma^2)T+\sigma\tilde{Z}(T)]}] \\ &= e^{-rT} S^a(0) e^{a(r-\delta-\frac{1}{2}\sigma^2)T} E^*[e^{a\sigma\tilde{Z}(T)}] \\ &= e^{-rT} S^a(0) e^{a(r-\delta-\frac{1}{2}\sigma^2)T} e^{\frac{1}{2}(a\sigma)^2 T} \end{aligned}$$

73. You are given:

(i) $\dfrac{dS(t)}{S(t)} = 0.3\,dt - \sigma\,dZ(t),$ $t \geq 0,$

where $Z(t)$ is a standard Brownian motion and σ is a positive constant.

(ii) There is a real number a such that

$$\frac{d[S(t)]^a}{[S(t)]^a} = -0.66\,dt + 0.6\,dZ(t), \quad t \geq 0.$$

Calculate σ.

(A) 0.16

(B) 0.20

(C) 0.27

(D) 0.60

(E) 1.60

Solution to (73)　　　　　　Answer: (B)

From (20.32), with σ changed to $-\sigma$, we obtain the following two equations:

　　　$a(-\sigma) = 0.6,$　　　　　　　　　　　　　　　　(1)

　　　$0.3a + \frac{1}{2}a(a-1)(-\sigma)^2 = -0.66.$　　　　　　(2)

From (1),

　　　$-\sigma = 0.6/a,$

substitution of which in (2) yields

　　　$0.3a + \frac{1}{2}a(a-1)(0.36/a^2) = -0.66,$

or

　　　$0 = 0.3a^2 + 0.84a - 0.18 = 0.02(5a - 1)(a + 3).$

Because σ is positive and because of (1), a must be negative. So $a = -3$, and we have

　　　$\sigma = -0.6/a = -0.6 / (-3) = 0.2.$

Remark: There is no need to memorize (20.32). Differentiate $[S(t)]^a$ using Itô's Lemma:

$$d[S(t)]^a = a[S(t)]^{a-1}dS(t) + \frac{1}{2}a(a-1)[S(t)]^{a-2}[dS(t)]^2 + 0$$
$$= [S(t)]^a\{a[0.3dt - \sigma dZ(t)] + \frac{1}{2}a(a-1)[0.3dt - \sigma dZ(t)]^2\}$$
$$= [S(t)]^a\{a[0.3dt - \sigma dZ(t)] + \frac{1}{2}a(a-1)\sigma^2 dt\},$$

or

$$\frac{d[S(t)]^a}{[S(t)]^a} = [0.3a + \frac{1}{2}a(a-1)\sigma^2]dt - a\sigma dZ(t),$$

comparing which with the equation in (ii) yields (1) and (2).

74. An equity-indexed product sold by an insurance company has a guarantee that is equivalent to a four-year at-the-money European put option on a nondividend-paying stock.

The company does not hedge its risk. Instead, the company will set aside a fund that accumulates at the risk-free rate.

The company wants the fund amount to be big enough so that there will be a 99% probability of being able to pay the guarantee out of the accumulated value of this fund.

You are given:

(i)   The current price of the stock is 40.

(ii)   The stock-price process is a geometric Brownian motion.

(iii)   The stock's volatility is 30%.

(iv)   The continuously compounded expected rate of return on the stock is 10%.

(v)   The effective annual risk-free interest rate is 2%.

Calculate the fund amount.

(A)   1

(B)   5

(C)   17

(D)   22

(E)   26

Solution to (74)   Answer: (E)

Let A denote the fund amount. Then, the condition for A is

$$\begin{aligned}
1\% &= \Pr\{[K - S(T)]_+ > e^{rT}A\} \\
&= \Pr\{[S(0) - S(4)]_+ > (1.02)^4 A\} \\
&= \Pr\{S(0) - S(4) > (1.02)^4 A\} \qquad \text{because } (1.02)^4 A \text{ is nonnegative} \\
&= \Pr\{S(0) - (1.02)^4 A > S(0)e^{(\alpha - \frac{1}{2}\sigma^2)\times 4 + \sigma Z(4)}\} \\
&= \Pr\{1 - (1.02)^4 A/S(0) > e^{0.22 + 0.3Z(4)}\} \\
&= \Pr\{1 - 0.02706A > e^{0.22 + 0.3\sqrt{4}Y}\} \qquad \text{where } Y \text{ is a normal } (0, 1) \text{ r.v.}
\end{aligned}$$

$$\Rightarrow \quad \frac{\ln(1 - 0.02706A) - 0.22}{0.6} = -2.326$$

$$\Rightarrow \quad \ln(1 - 0.02706A) = -1.3956 + 0.22 = -1.1756$$

$$\Rightarrow \quad A = (1 - e^{-1.1756})/0.02706 = 25.549 \approx 26$$

Remarks: (i) An alternative solution is to use (18.23) on page 599.

$$1\% = \Pr\{S(4) < S(0) - 1.02^4 A\} = N(-\hat{d}_2),$$

where $\hat{d}_2 = \dfrac{\ln\dfrac{S(0)}{S(0) - 1.02^4 A} + (0.1 - \dfrac{0.3^2}{2})\times 4}{0.3\sqrt{4}}$.

Since $N(-2.326) = 1\%$,

$$\frac{\ln\dfrac{S(0)}{S(0) - 1.02^4 A} + (0.1 - \dfrac{0.3^2}{2})\times 4}{0.3\sqrt{4}} = 2.326$$

$$\ln\frac{40}{40 - 1.02^4 A} = 1.1756$$

$$A = 25.549 \approx 26.$$

(ii) *Value at Risk* is discussed in McDonald's Chapter 25, which is not part of the Exam MFE/3F syllabus. For the put option, the quantity $(1.02)^4 A$ can be viewed as the four-year Value at Risk with a 99% level of confidence.

(iii) Suppose that the company employs the following *static hedging* strategy. Upon the sale of a policy, it would sell short a fraction of a share of the stock; there is no rebalancing. Then, the problem is to find A such that

$$\begin{aligned}
1\% &= \Pr\{[S(0) - S(T)]_+ + \varphi S(T) > e^{rT}[A + \varphi S(0)]\} \\
&= \Pr\{[S(0) - S(T)]_+ > e^{rT}A + \varphi[e^{rT}S(0) - S(T)]\},
\end{aligned}$$

which is a harder problem. Because the right-hand side of the last inequality can be negative, we cannot simply replace $[S(0) - S(T)]_+$ by $[S(0) - S(T)]$.

(iv) Suppose that the exercise price is $K = S(0)e^{rT}$, instead of $K = S(0)$. Then
$$F_{0,T}^P(K) = e^{-rT}K = S(0) = F_{0,T}^P(S).$$
It follows that the European put price and European call price are the same; the price is
$$S(0)[N(\tfrac{1}{2}\sigma\sqrt{T}) - N(-\tfrac{1}{2}\sigma\sqrt{T})] = S(0)[2N(\tfrac{1}{2}\sigma\sqrt{T}) - 1],$$
which is a nontrivial fraction of $S(0)$ if $\sigma\sqrt{T}$ is not small. On the other hand, because
$$[S(0)e^{rT} - S(T)] = S(0)[e^{rT} - e^{(\alpha - \frac{1}{2}\sigma^2)T + \sigma Z(T)}],$$
we have
$$\Pr[S(0)e^{rT} - S(T) > 0] = \Pr[rT > (\alpha - \tfrac{1}{2}\sigma^2)T + \sigma Z(T)] = N\left(\frac{r - (\alpha - \frac{1}{2}\sigma^2)}{\sigma}T\right),$$
which is a formula that can also be found on page 603 of McDonald (2006). Thus, if $r < \alpha - \tfrac{1}{2}\sigma^2$ (which is not an unreasonable assumption), the "true" probability that a European put option with exercise price $K = S(0)e^{rT}$ and a long maturity date T will be in the money is small. Hence, some would argue that such a put option should cost little, contradicting the fact that its price is a substantial fraction of $S(0)$.

75. You are using Monte Carlo simulation to estimate the price of an option X, for which there is no pricing formula. To reduce the variance of the estimate, you use the control variate method with another option Y, which has a pricing formula.

You are given:

(i) The naive Monte Carlo estimate of the price of X has standard deviation 5.

(ii) The same Monte Carlo trials are used to estimate the price of Y.

(iii) The correlation coefficient between the estimated price of X and that of Y is 0.8.

Calculate the minimum variance of the estimated price of X, with Y being the control variate.

(A)   1.0

(B)   1.8

(C)   4.0

(D)   9.0

(E)   16.0

Solution to (75)   Answer: (D)

Following (19.10), we let $X^* = X_{sim} + \beta(Y_{true} - Y_{sim})$. Its variance is

$$\mathrm{Var}(X^*) = \mathrm{Var}(X_{sim}) + \beta^2 \, \mathrm{Var}(Y_{sim}) - 2\beta \, \mathrm{Cov}(X_{sim}, Y_{sim}),$$

which is (19.11).

To find the optimal β, differentiate the RHS with respect to β and equate the derivative to 0. The solution of the resulting equation is the optimal β, which is

$$\frac{\mathrm{Cov}(X_{sim}, Y_{sim})}{\mathrm{Var}(Y_{sim})},$$

a result that can be found on page 632 in McDonald (2006).

Thus, that the minimum of $\mathrm{Var}(X^*)$ is

$$\mathrm{Var}(X_{sim}) + \left[\frac{\mathrm{Cov}(X_{sim}, Y_{sim})}{\mathrm{Var}(Y_{sim})}\right]^2 \mathrm{Var}(Y_{sim}) - 2\frac{\mathrm{Cov}(X_{sim}, Y_{sim})}{\mathrm{Var}(Y_{sim})}\mathrm{Cov}(X_{sim}, Y_{sim})$$

$$= \mathrm{Var}(X_{sim}) - \frac{[\mathrm{Cov}(X_{sim}, Y_{sim})]^2}{\mathrm{Var}(Y_{sim})}$$

$$= \mathrm{Var}(X_{sim}) \left\{1 - \frac{[\mathrm{Cov}(X_{sim}, Y_{sim})]^2}{\mathrm{Var}(X_{sim})\mathrm{Var}(Y_{sim})}\right\}$$

$$= \mathrm{Var}(X_{sim})\{1 - [\mathrm{Corr}(X_{sim}, Y_{sim})]^2\}$$

$$= 5^2(1 - 0.8^2)$$

$$= 9.$$

Remarks: (i) For students who have learned the concept of *inner product* (*scalar product* or *dot product*), here is a way to view the problem. Given two vectors x and y, we want to minimize the length $\|x - \beta y\|$ by varying β. To find the optimal β, we differentiate $\|x - \beta y\|^2$ with respect to β and equate the derivative with 0. The optimal β is $\langle x, y\rangle/\|y\|^2$. Hence,

$$\underset{\beta}{\text{Minimum}} \|x - \beta y\|^2 = \left\|x - \frac{\langle x, y\rangle}{\|y\|^2}y\right\|^2 = \|x\|^2\left(1 - \frac{\langle x, y\rangle^2}{\|x\|^2\|y\|^2}\right).$$

(ii) The quantity $\dfrac{\langle x, y\rangle}{\|x\|\cdot\|y\|}$ is the cosine of the angle between the vectors x and y.

(iii) A related formula is (4.4) in McDonald (2006).

76. You are given the following information about a Black-Derman-Toy binomial tree modeling the movements of effective annual interest rates:

(i) The length of each period is one year.

(ii) In the first year, $r_0 = 9\%$.

(iii) In second year, $r_u = 12.6\%$ and $r_d = 9.3\%$

(iv) In third year, $r_{uu} = 17.2\%$ and $r_{dd} = 10.6\%$. The value of r_{ud} is not provided.

Calculate the price of a 3-year interest-rate cap for notional amount 10,000 and cap rate 11.5%.

(A) 202

(B) 207

(C) 212

(D) 217

(E) 222

Solution to (76)   Answer: (D)

A related problem is #15 in this set of sample questions and solutions.

Caps are usually defined so that the initial rate, r_0, even if it is greater than the cap rate, does not lead to a payoff, i.e., the year-1 caplet is disregarded. In any case, the year-1 caplet in this problem has zero value because r_0 is lower than the cap rate.
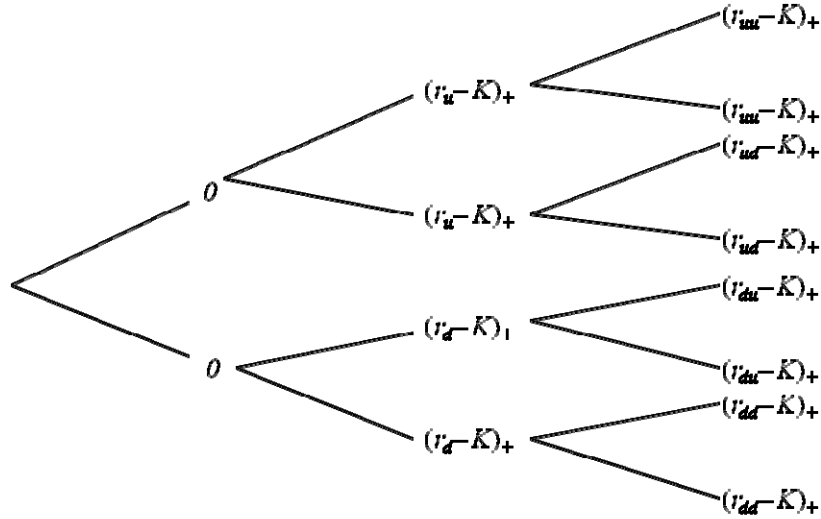
Since a 3-year cap is the sum of a year-2 caplet and a year-3 caplet, one way to price a 3-year cap is to price each of the two caplets and then add up the two prices. However, because the payoffs or cashflows of a cap are not path-dependent, the method of *backward induction* can be applied, which is what we do next.

It seems more instructive if we do not assume that the binomial tree is recombining, i.e., we do not assume $r_{ud} = r_{du}$. Thus we have the following three-period (three-year) interest rate tree.
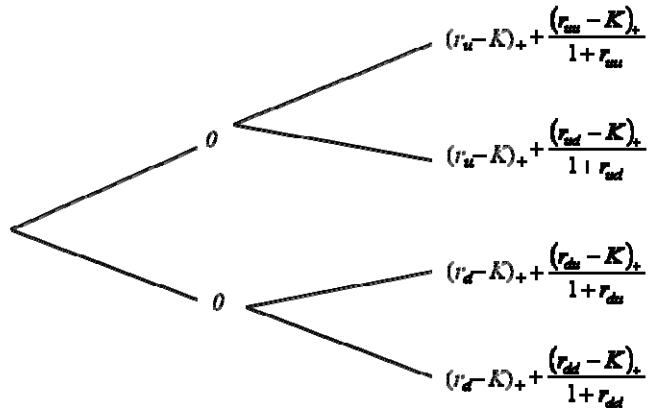

We also do not assume the risk-neutral probabilities to be ½ and ½. We use p^* to denote the risk-neutral probability of an up move, and q^* the probability of a down move.
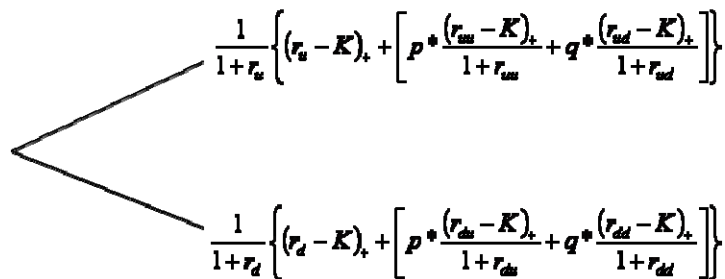
In the next diagram, we show the payoffs or cashflows of a 3-year interest-rate cap for notional amount 1 and cap rate K. Here, $(r - K)_+$ means $\max(0, r - K)$.

Discounting and averaging the cashflows at time 3 back to time 2:

Moving back from time 2 back to time 1:

$$\frac{1}{1+r_u}\left\{(r_u - K)_+ + \left[p * \frac{(r_{uu} - K)_+}{1+r_{uu}} + q * \frac{(r_{ud} - K)_+}{1+r_{ud}}\right]\right\}$$

$$\frac{1}{1+r_d}\left\{(r_d - K)_+ + \left[p * \frac{(r_{du} - K)_+}{1+r_{du}} + q * \frac{(r_{dd} - K)_+}{1+r_{dd}}\right]\right\}$$

Finally, we have the time-0 price of the 3-year interest-rate cap for notional amount 1 and cap rate K:

$$\frac{1}{1+r_0}\left\{ p*\frac{1}{1+r_u}\left[(r_u-K)_+ + p*\frac{(r_{uu}-K)_+}{1+r_{uu}} + q*\frac{(r_{ud}-K)_+}{1+r_{ud}} \right] \right.$$
$$\left. + q*\frac{1}{1+r_d}\left[(r_d-K)_+ + p*\frac{(r_{du}-K)_+}{1+r_{du}} + q*\frac{(r_{dd}-K)_+}{1+r_{dd}} \right] \right\}. \tag{1}$$

As we mentioned earlier, the price of a cap can also be calculated as the sum of caplet prices. The time-0 price of a year-2 caplet is

$$\frac{1}{1+r_0}\left[p*\frac{(r_u-K)_+}{1+r_u} + q*\frac{(r_d-K)_+}{1+r_d} \right].$$

The time-0 price of a year-3 caplet is

$$\frac{1}{1+r_0}\left\{ p*\frac{1}{1+r_u}\left[p*\frac{(r_{uu}-K)_+}{1+r_{uu}} + q*\frac{(r_{ud}-K)_+}{1+r_{ud}} \right] + q*\frac{1}{1+r_d}\left[p*\frac{(r_{du}-K)_+}{1+r_{du}} + q*\frac{(r_{dd}-K)_+}{1+r_{dd}} \right] \right\}. \tag{2}$$

It is easy to check that the sum of these two caplet pricing formulas is the same as expression (1).

Rewriting expression (2) as

$$(p*)^2\frac{(r_{uu}-K)_+}{(1+r_0)(1+r_u)(1+r_{uu})} + p*q*\frac{(r_{ud}-K)_+}{(1+r_0)(1+r_u)(1+r_{ud})}$$
$$+q*p*\frac{(r_{du}-K)_+}{(1+r_0)(1+r_d)(1+r_{du})} + (q*)^2\frac{(r_{du}-K)_+}{(1+r_0)(1+r_d)(1+r_{dd})}$$

shows the path-by-path nature of the year-3 caplet price.

A Black-Derman-Toy interest rate tree is a recombining tree (hence $r_{ud}=r_{du}$) with $p*=q*=\frac{1}{2}$. Expression (1) simplifies as
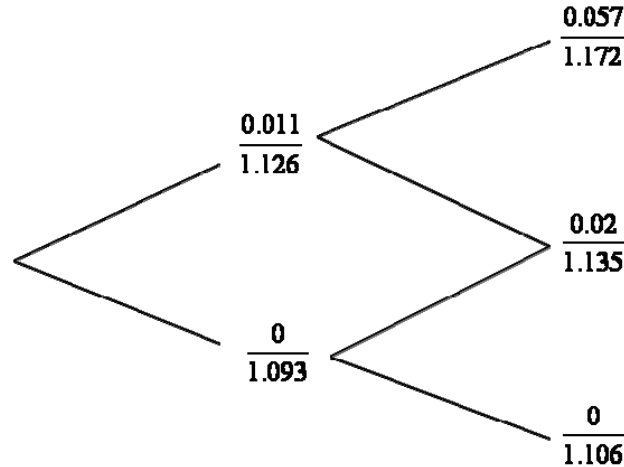
$$\frac{1}{2}\frac{1}{1+r_0}\left\{ \frac{1}{1+r_u}\left[(r_u-K)_+ + \frac{1}{2}\left(\frac{(r_{uu}-K)_+}{1+r_{uu}} + \frac{(r_{ud}-K)_+}{1+r_{ud}} \right) \right] \right.$$
$$\left. + \frac{1}{1+r_d}\left[(r_d-K)_+ + \frac{1}{2}\left(\frac{(r_{ud}-K)_+}{1+r_{ud}} + \frac{(r_{dd}-K)_+}{1+r_{dd}} \right) \right] \right\}. \tag{3}$$

In this problem, the value of r_{ud} is not given. In each period of a B-D-T model, the interest rates in different states are terms of a geometric progression. Thus, we have
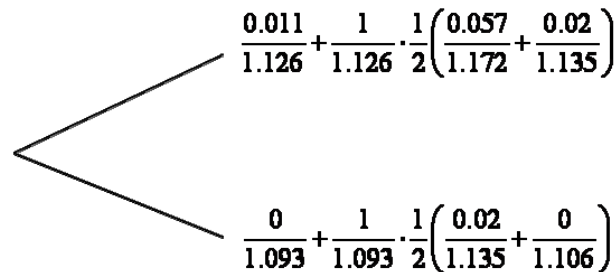
$$0.172/r_{ud} = r_{ud}/0.106,$$

from which we obtain $r_{ud} = 0.135$. With this value, we can price the cap using (3).

Instead of using (3), we now solve the problem directly. As in Figure 24.9 on page 806 of McDonald (2006), we first discount each cap payment to the beginning of the payment year. The following tree is for notational amount of 1 (and $K = 0.115$).

Discounting and averaging the cashflows at time 2 back to time 1:

Thus the time-0 price of the cap is

$$\frac{1}{1.09} \cdot \frac{1}{2} \left\{ \frac{1}{1.126} \left[0.011 + \frac{1}{2} \left(\frac{0.057}{1.172} + \frac{0.02}{1.135} \right) \right] + \frac{1}{1.093} \left[0 + \frac{1}{2} \left(\frac{0.02}{1.135} + \frac{0}{1.106} \right) \right] \right\}$$

$$= \frac{1}{1.09} \cdot \frac{1}{2} \left\{ \frac{1}{1.126} \times 0.044128 + \frac{1}{1.093} \times 0.008811 \right\}$$

$$= 0.02167474.$$

Answer (D) is correct, because the notation amount is 10,000.

Remark: The prices of the two caplets for notional amount 10,000 and $K = 0.115$ are 44.81 and 171.94. The sum of these two prices is 216.75.