

Derivatives and Their Portfolios, Put-Call Parity

- Review: (European derivatives)

<i>Derivative</i>	forward	call option	put option
<i>payoffs</i>	$S_T - F_{0,T}(S)$	$(S_T - K)^+$	$(K - S_T)^+$

- Long and short positions

- $F_{0,T}(S) = e^{(r-\delta)T} S_0$, $F_{0,T}^P(S) = e^{-\delta T} S_0$

- Payoff diagrams

The (time T) payoffs of a basic derivative as a function of $S(T)$, the spot asset price at the expiration time T , is continuous and piecewise linear.

- Synthetic forward and its payoff

- No-arbitrage principle, one price principle

A working version of the one price principle is

Let $A(t)$ and $B(t)$ be the time t values of two assets A and B respectively. Then

$$A(T) = B(T) \text{ for some } T > 0 \implies A(0) = B(0)$$

- A portfolio is a linear combination of basic derivatives (forward, call, put) and the asset itself. Since the payoff of any basic derivative or the asset itself is continuous and piecewise linear, *the payoff of a portfolio is continuous and piecewise linear.*

- EXAMPLE. Problem 2 form the samples. (Using the payoff diagrams.)

- Put-call Parity

Consider the following two portfolios containing European derivatives with the same expiration time T :

- Portfolio 1: a long position in a call and a short position in a put, both have the same strike K ;
- Portfolio 2: a prepaid forward on the stock and borrowing $PV_{0,T}(K)$.

The time T payoffs of both portfolios are $S(T) - K$. No Arbitrage Principle/One Price Principle implies that they have the same time 0 values. Therefore,

$$C(K, T) - P(K, T) = F_{0,T}^P(S) - PV_{0,T}(K) \quad (1)$$

Viewing $PV_{0,T}(K)$ as $F_{0,T}^P(K)$, the above equation becomes

$$C(K, T) - P(K, T) = F_{0,T}^P(S) - F_{0,T}^P(K) \quad (2)$$

- Versions of Put-Call

- For continuous dividend rate δ and continuous interest rate r , $F_{0,T}^P(S) = e^{-\delta T} S_0$ and $PV_{0,T}(K) = e^{-rT} K$, (1) $\implies$

$$C(K, T) - P(K, T) = e^{-\delta T} S_0 - e^{-rT} K$$

- For discrete dividend, $F_{0,T}^P(S) = S_0 - \text{PV}(\text{Div})$, (1) $\Rightarrow$

$$C(K, T) - P(K, T) = S_0 - \text{PV}(\text{Div}) - e^{-rT}K$$

- For annual effective interest rate i , use $(1+i)^{-T}K$ for $\text{PV}_{0,T}(K)$ above.
- If S is a foreign currency, say euro, then $\delta = r_{\text{euro}}$ (interest rate in euro) and $S_0 = x_0$ (spot exchange rate in dollars per euro), (1) becomes

$$C(K, T) - P(K, T) = e^{-r_{\text{euro}}T}x_0 - e^{-r_{\text{dollar}}T}K$$

- General Parity and Exchange Option

Instead of paying money (K dollars), if we use asset Q , the strike asset, to exchange for stock S , (2) becomes

$$C(S, Q, T) - P(S, Q, T) = F_{0,T}^P(S) - F_{0,T}^P(Q)$$

or more general, at a time t between 0 and T ,

$$C(S, Q, T-t) - P(S, Q, T-t) = F_{t,T}^P(S) - F_{t,T}^P(Q)$$

Especially at time T ($t = T$), $C(S, Q, 0) = (S_T - Q_T)^+$, $P(S, Q, 0) = (Q_T - S_T)^+$, and $C(S, Q, 0) - P(S, Q, 0) = S_T - Q_T$.

- Perspectives on Buy/Sell, Call/Put

Transactions	Options	Payoffs (at time T)
Buy stock, pay dollars	Call option on stock with strike K	$(S_T - K)^+$
Sell dollars, receive stock	Put option on $\$K$ with strike S	$(S_T - K)^+$

Same payoff $(S_T - K)^+$ at time $T \Rightarrow$ same option value at time 0 (or at any $t < T$).

Example. Currency Options

Suppose the current exchange rate of dollar to euro is x_0 dollars per euro (that is, you can “buy” one euro in $\$x_0$), or x_0 $\$/\text{€}$. (x_0 is the S_0 in stock case.)

Consider two options with the same expiration time T :

- a call option on one € with strike of K dollars
- a put option on K dollars with strike of one €

By the above discussion, these two options have the same value now. Furthermore, one put option on K dollars with strike of one € can be viewed as K put options on 1 dollar with strike of $(\frac{1}{K}) \text{€}$. Put them side by side:

Options	One call option on 1 € with strike price K	One put option on K dollars with strike of 1 € $\cong$ K put options on 1 dollar with strike of $(\frac{1}{K})\text{€}$
Time 0 Prices	$C_{\$}(K, T)$ in $\text{\$}$	$K \cdot P_{\text{€}}(\frac{1}{K}, T)$ in €

Using the current exchange rate x_0 to convert euro to dollar, we have

$$C_{\$}(K, T) = x_0 K \cdot P_{\text{€}}\left(\frac{1}{K}, T\right) \quad (\text{in dollars})$$

- European vs. American

- Relations on Prices

$$S \geq C_{\text{Amer}}(S, K, T) \geq C_{\text{Eur}}(S, K, T) \geq (\text{PV}_{0,T}(F_{0,T}) - \text{PV}_{0,T}(K))^+$$

$$K \geq P_{\text{Amer}}(S, K, T) \geq P_{\text{Eur}}(S, K, T) \geq (\text{PV}_{0,T}(K) - \text{PV}_{0,T}(F_{0,T}))^+$$

- Early Exercise

- * *Never early exercise an American call on a nondividend-paying stock* because

$$\begin{aligned} C_{\text{Amer}}(K, T - t) &\geq C_{\text{Eur}}(K, T - t) = S_t - e^{-r(T-t)}K + P_{\text{Eur}}(K, T - t) \\ &= S_t - K + (1 - e^{-r(T-t)})K + P_{\text{Eur}}(K, T - t) \\ &> S_t - K \end{aligned}$$

This means that $C_{\text{Amer}} = C_{\text{Eur}}$ on nondividend-paying stocks.

- * For American call on dividend-paying stocks and American put, early-exercise may have advantage.

- Time to Expiration

- Strike Prices

Viewed as functions of strike price K ,

- * $C(K, T)$ is decreasing in K ; $P(K, T)$ is increasing in K : if $K_1 < K_2$,

$$C(K_1, T) \geq C(K_2, T), \quad P(K_1, T) \leq P(K_2, T)$$

- * Slope is less than 1 (in absolute value): if $K_1 < K_2$,

$$\left| \frac{C(K_2) - C(K_1)}{K_2 - K_1} \right| \leq 1, \quad \left| \frac{P(K_2) - P(K_1)}{K_2 - K_1} \right| \leq 1$$

$$\Rightarrow C(K_1) - C(K_2) \leq K_2 - K_1, \quad P(K_2) - P(K_1) \leq K_2 - K_1$$

that is the change in price is less than the change in strike.

- * C and P are convex: if $K_1 < K_2 < K_3$,

$$\frac{C(K_2) - C(K_1)}{K_2 - K_1} \leq \frac{C(K_3) - C(K_2)}{K_3 - K_2}, \quad \frac{P(K_2) - P(K_1)}{K_2 - K_1} \leq \frac{P(K_3) - P(K_2)}{K_3 - K_2}$$

$$\text{For } C, \text{ it can be written as } \frac{C(K_1) - C(K_2)}{K_2 - K_1} \geq \frac{C(K_2) - C(K_3)}{K_3 - K_2}$$