Copula Functions and their Application in Pricing and Risk Managing Multiname Credit Derivative Products

by

Stefano S. Galiani

Department of Mathematics
King's College London
The Strand, London WC2Y 2LS
United Kingdom

Email: stefanogaliani@hotmail.com
Tel: +44 7816 994 902

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Abstract

This paper presents and numerically implements a methodology to price credit derivative products referencing a portfolio of underlying assets. We develop a copula based framework to model the default dependency among obligors and offer algorithms for pricing Basket Default Swaps and Collateralized Debt Obligations. A risk neutral methodology by which calibrate copula parameters to market quotes is taken into account, and different methods of calibration are illustrated and implemented. We numerically calculate the sensitivity of prices to the main state variables affecting their values, namely recovery rates, default correlation and credit quality of the underlying portfolio. By assuming two alternative specifications (Gaussian and Student's t) of the copula function used to describe the joint distribution of default times among obligors, we then demonstrate the effect of the asymptotic tail dependence on modeling portfolio defaults and losses.

Keywords: Basket default swap, Collateralized debt obligation, Credit default swap, Copula functions, Maximum likelihood estimation, Inference functions for margins, Canonical maximum likelihood, Tail dependence, Default times, Simulation algorithm, Hazard rates, Loss distribution.

Contents

1 Introduction

The rapidly growing credit derivatives market allows corporates and financial institutions to efficiently transfer and manage their credit exposures by way of an ever-increasing spectrum of instruments. Multiple underlyings products, with payoffs contingent on credit event realizations of a specified portfolio of obligors (reference entities), are among those assuming a relevant role. A typical example is given by Basket Default Swaps, with a payment (made by the protection seller and usually equal to the net loss[1] on the defaulted security) contingent upon the time of default of a particular combination[2] of the credits referenced by the contract. On the other side, the protection buyer will correspond periodic payments to the other party either until the legal maturity of the contract or earlier, in case the specified combination of trigger events occurs. Another important financial product is represented by the Collateralized Debt Obligations (CDO), which are securities, backed by a portfolio of assets (e.g. sovereign or corporate bonds, loans, residential and commercial mortgages), issued and tranched into pieces (generally senior, mezzanine and equity tranche) of different subordination level corresponding to specific portfolio losses. In this way losses in the collateral pool (either on interests flow or principal redemption) are first absorbed by the lowest tranche and then, according to the severity of losses, up to the next tranches, thus enabling originators and investors to create a set of customized return-risk profiles. Given the multiple-underlying nature of these structures, modeling dependence among different obligors included in a generic multi-name credit derivative product is undoubtedly one of the crucial parts of any pricing model and successful risk managing strategy. To this end copula functions offer an efficient and consistent approach to analyze and model the correlated default timing of a credit sensitive portfolio. The aim of this paper is to present a copula based framework by which analyze multiname credit derivative products. The paper is structured in seven sections. In the next section we introduce copula functions and establish some useful properties. § 3 will present a risk neutral methodology to extract single name default probabilities from market quotes. §§ 4 and 5 will present a copula based simulation procedure for evaluating Basket Default Swaps and CDOs,

[1] The net loss is defined to be equal to the notional amount of the contract times $(1-Rec)$ where Rec stands for the recovery (percentage value of the defaulted asset) rate.

[2] Generally in the market we have the following combinations: first or second obligor to default, first m out of n obligors to default or last m out of n obligors to default.

assessing the influence of different price drivers (correlation, recovery rates, credit worthiness and size of the underlying portfolio) on modeling portfolio defaults and losses. § 6 concludes and two formal proofs are given in the appendix in § 7.

2 Copula Functions

The analysis of copula functions is organized as follows: starting from the approach suggested by Nelsen [28], in § 2.1 we will review the basic definitions and properties of copula functions. § 2.2 will present some of the most representative classes of copula functions, and their main characteristics in terms of density functions (for elliptical copulae) and generator functions (for Archimedean copulas). § 2.3 will introduce the concept of tail dependence, which will be further analyzed with regards to the two most widely used elliptical copulae, the gaussian and the t copula. § 2.4 will conclude with the problem of fitting copula parameters to market data. Different approaches will be reviewed and compared in terms of computational efficiency.

2.1 Definition and basic properties of copula functions

Definition 1 *An n-dimensional copula is a function $C : [0,1]^n \rightarrow [0,1]$ which has the following properties:*

1. $C(\mathbf{u})$ is increasing in each component $\mathbf{u}_k$ with $k \in \{1, 2, \ldots, n\}$.
2. For every vector $\mathbf{u} \in [0,1]^n$, $C(\mathbf{u}) = \mathbf{0}$ if at least one coordinate of the vector $\mathbf{u}$ is 0 and $C(\mathbf{u}) = \mathbf{u}_k$ if all the coordinate of $\mathbf{u}$ are equal to 1 except the k-th one.
3. For every $\mathbf{a}$,$\mathbf{b} \in [0,1]^n$ with $\mathbf{a} \leq \mathbf{b}$, given a hypercube $\mathbf{B} = [\mathbf{a}, \mathbf{b}] = [a_1, b_1] \times [a_2, b_2] \cdots \times [a_n, b_n]$ whose vertices lie in the domain of C, its volume[3] $V_C(B) \geq 0$.

[3] The volume $V_C(B)$ of a n-box $B = [\mathbf{a}, \mathbf{b}]$, is defined as follows:

$$\begin{aligned} V_C(B) &= \sum_{\mathbf{d}} sgn(\mathbf{d})\, C(\mathbf{d}) \\ &= \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+i_2+\cdots i_n} C(d_{1i_1}, d_{2i_2}, \ldots, d_{ni_n,}) \geq 0 \end{aligned}$$

This definition shows that C is a multivariate distribution function with uniformly distributed margins. The statistical interpretation of the above properties will become further meaningful once we will adapt the definition of copula functions to a vector of random variables. But before this, we need an auxiliary theorem, which constitutes one the most relevant result in the copula framework.

Theorem 2 *(Sklar) Let G be an n-dimensional distribution function with margins $F_1, F_2, \ldots, F_n$. Then there exist an n-dimensional copula C such that, for $\mathbf{x} \in \mathbb{R}^n$ we have*

$$G(x_1, x_2, \ldots, x_n) = C\left(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)\right). \tag{1}$$

Moreover, if $F_1, F_2, \ldots, F_n$ are continuous, then C is unique.

Sklar's theorem expresses the basic idea of dependence modeling via copula functions, by stating that for any multivariate distribution function, the univariate margins (the distribution functions in case of random variables) and the dependence structure can be separated, with the latter completely described by a copula function.

As pointed out in Scaillet [31], Sklar's theorem has an important corollary:

Corollary 3 *Let G and C be, respectively, an n-dimensional distribution function (with continuous univariate margins $F_1, F_2, \ldots F_n$) and an n-dimensional copula function. Then, for any $\mathbf{u} \in [0,1]^n$, we have*

$$C\left(u_1, u_2, \ldots, u_n\right) = G(F_1^{-1}(u_1), F_2^{-1}(u_2), \ldots, F_n^{-1}(u_n)), \tag{2}$$

where $F_i^{-1}(u_i)$ denotes the inverse of the cumulative distribution function, namely, for $u_i \in [0,1]$, $F_i^{-1}(u_i) = \inf\{x : F_i(x) \geq u_i\}$. The importance of equation (2) will be clear in the following sections once we will present a general framework for simulation of random numbers generated by a specific copula. Now, let $(X_1, X_2, \ldots, X_n)'$ be an n-dimensional vector of random variables with distribution functions $(F_1, F_2, \ldots, F_n)$ and joint distribution function G. Then, by Sklar's theorem, if $(F_1, F_2, \ldots, F_n)$ are continuous functions, $(X_1, X_2, \ldots, X_n)'$ has a unique copula as described by the following representation

$$G(x_1, x_2, \ldots, x_n) = \mathbb{P}\left(X_1 \leq x_1, \ldots, X_n \leq x_n\right) = C\left(F_1(x_1), \ldots, F_n(x_n)\right).$$

where $d_{j1} = a_j$ and $d_{j2} = b_j$ for all $j \in \{1, 2, \ldots, n\}$.

This representation of copula functions, allows us to further re-establish the last two properties of definition 1. In fact, property (2) follows from the fact that, by the so called "probability-integral transform" (see, Casella & Berger [5], p. 54), if the random variables X and Y have continuous distribution function F_X and F_Y, then the random variables $U = F_X(X)$ and $V = F_Y(Y)$ are uniformly distributed on $[0, 1]$ and therefore, in the bivariate case,

$$C(u, 1) = \mathbb{P}(U \leq u, V \leq 1) = \mathbb{P}(U \leq u) = u \tag{3}$$

and

$$C(u, 0) = \mathbb{P}(U \leq u, V \leq 0) = 0.$$

Property (3) ensures that the copula function C respects the defining characteristic of a proper multivariate distribution function, assigning non-negative weights to all rectangular subsets in $[0, 1]^n$.

Remark 1 *By applying Sklar's theorem and by exploiting the relation between the distribution and the density function*[4]*, we can easily derive the multivariate copula density* $c(F_1(x_1), \dots, F_n(x_n))$ *associated with a copula function* $C(F_1(x_1), \dots, F_n(x_n))$*:*

$$\begin{aligned} f(x_1, \dots, x_n) &= \frac{\partial^n [C(F_1(x_1), \dots, F_n(x_n))]}{\partial F_1(x_1) \dots \partial F_n(x_n)} \cdot \prod_{i=1}^{n} f_i(x_i) \\ &= c(F_1(x_1), \dots, F_n(x_n)) \cdot \prod_{i=1}^{n} f_i(x_i), \end{aligned}$$

where we define

$$c(F_1(x_1), \dots, F_n(x_n)) = \frac{f(x_1, \dots, x_n)}{\prod_{i=1}^{n} f_i(x_i)}. \tag{4}$$

As we will see in § 2.4, knowledge of the associated copula density will be particularly useful in order to calibrate its parameters to real market data.

[4] As known, in the univariate case, the density function $f(x)$ of a random variable X can be obtained by the cumulative distribution function via the following relation:

$$f(x) = \frac{\partial F(x)}{\partial x}.$$

2.2 Classification of Copula Functions

In this section we will present some of the families of copula functions. With regard to classification purposes and, more importantly, in view of our subsequent applications , greater details will be provided for copulae belonging to the elliptical family[5], namely the gaussian and the t copula.

2.2.1 Multivariate Gaussian Copula

Definition 4 *Let R be a symmetric, positive definite matrix with $diag(R) = \mathbf{1}$ and let Φ_R the standardized multivariate normal distribution with correlation matrix R[6]. Then the Multivariate Gaussian Copula is defined as*

$$C(u_1, u_2, \ldots, u_n; R) = \Phi_R\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots, \Phi^{-1}(u_n)\right), \tag{5}$$

where $\Phi^{-1}(u)$ denotes the inverse of the normal cumulative distribution function. The associated multinormal copula density is obtained by applying equation (4):

$$\begin{aligned} c(\Phi(x_1), \ldots, \Phi(x_n)) &= \frac{f^{gaussian}(x_1, \ldots, x_n)}{\prod_{i=1}^n f_i^{gaussian}(x_i)} \\ &= \frac{\frac{1}{(2\pi)^{\frac{n}{2}}|R|^{\frac{1}{2}}}\exp(-\frac{1}{2}\mathbf{x}'R^{-1}\mathbf{x})}{\prod_{i=1}^n \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}x_i^2)}, \end{aligned}$$

and hence, fixing $u_i = \Phi(x_i)$, and denoting with $\zeta = (\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n))'$ the vector of the gaussian univariate inverse distribution functions, we have

$$c(u_1, u_2, \ldots, u_n; R) = \frac{1}{|R|^{\frac{1}{2}}}\exp\left[-\frac{1}{2}\zeta'\left(R^{-1} - I\right)\zeta\right]. \tag{6}$$

[5]Based on the definition of Fang, Kotz & Ng [11], if $\mathbf{X}$ is a n-dimensional vector of random variables and, for some $\mu \in \mathbb{R}^n$ and some $n \times n$ nonnegative definite, symmetric matrix Σ, the characteristic function $\varphi_{\mathbf{X}-\mu}(\mathbf{t})$ of $\mathbf{X} - \mu$ is a function of the quadratic form $\mathbf{t}^T\Sigma\mathbf{t}$, then $\mathbf{X}$ has an **elliptical distribution** with parameters (μ, Σ, φ), and we write $\mathbf{X} \sim \mathbf{E}_n(\mu, \Sigma, \varphi)$.

[6]Given a random vector $\mathbf{X} = (X_1, \ldots, X_n)'$ we define the standardized normal joint density function $f(\mathbf{x})$ with correlation matrix R, as follows:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}|R|^{\frac{1}{2}}}\exp(-\frac{1}{2}\mathbf{x}'R^{-1}\mathbf{x})$$

Figure 1 shows the surface of the gaussian copula density as depicted in equation (6) for the bivariate case with correlation r.

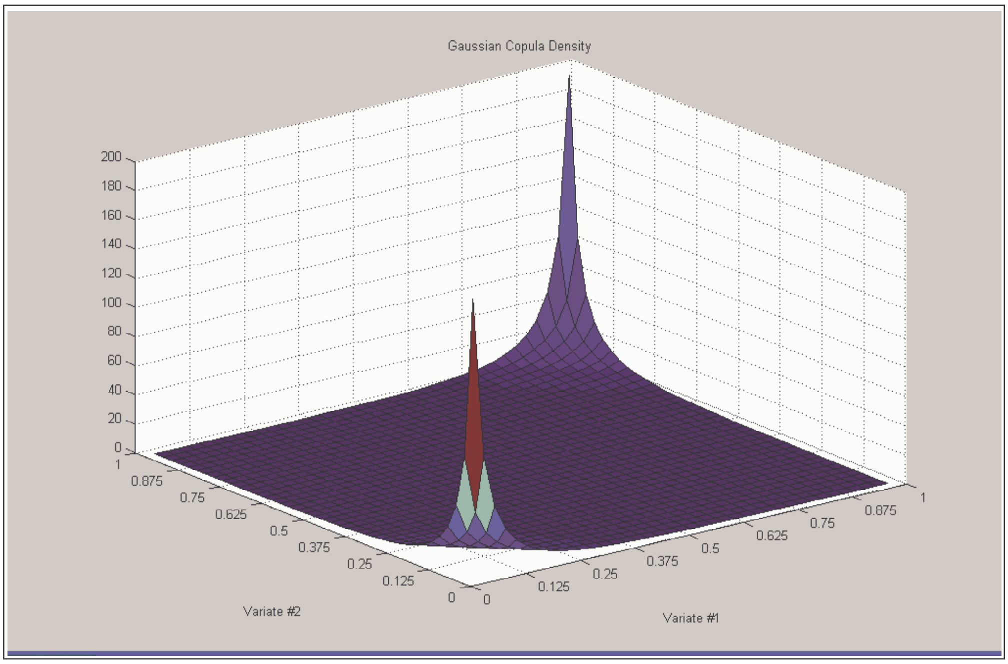

Figure 1. Gaussian copula density, r=0.5

2.2.2 Multivariate Student's t Copula

Definition 5 *Let R be a symmetric, positive definite matrix with $diag(R) = \mathbf{1}$ and let $T_{R,\nu}$ the standardized multivariate Student's t distribution with correlation matrix R and ν degrees of freedom*[7]*. Then the Multivariate Student's t Copula is defined as follows*

$$C(u_1, u_2, \ldots, u_n; R, \nu) = T_{R,\nu}\left(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2), \ldots, t_\nu^{-1}(u_n)\right), \tag{7}$$

where $t_\nu^{-1}(u)$ denotes the inverse of the Student's t cumulative distribution function. The associated Student's t copula density is obtained by applying again equation (4):

$$\begin{aligned} c(u_1, u_2, \ldots, u_n; R, \nu) &= \frac{f^{Student}(x_1, \ldots, x_n)}{\prod_{i=1}^n f_i^{Student}(x_i)} \\ &= |R|^{-\frac{1}{2}} \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left[\frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)}\right]^n \frac{(1+\frac{\zeta' R^{-1}\zeta}{\nu})^{-\frac{\nu+n}{2}}}{\prod_{i=1}^n (1+\frac{\zeta_i^2}{\nu})^{-\frac{\nu+1}{2}}}, \end{aligned} \tag{8}$$

where $\zeta = (t_\nu^{-1}(u_1), t_\nu^{-1}(u_2), \ldots, t_\nu^{-1}(u_n))'$.

Figure 2 shows the surface of the Student's t copula density as depicted in equation (8) for the bivariate case with correlation r.

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[7] Following Johnson & Kotz [17], p. 134, given a random vector $\mathbf{X} = (X_1, \ldots, X_n)'$ with a joint standardized multinormal distribution with correlation matrix R and a χ_ν^2-distribuited random variable S, independent from $\mathbf{X}$, we define the standardized multivariate Student's t joint density function with correlation matrix R and ν degrees of freedom, as the joint distribution function of the random vector $\mathbf{Y} = \left(\frac{X_1}{S/\sqrt{\nu}}, \ldots, \frac{X_n}{S/\sqrt{\nu}}\right)'$:

$$f(\mathbf{y}) = \frac{\Gamma\left[\frac{1}{2}(\nu+n)\right]}{\Gamma\left(\frac{1}{2}\nu\right)} \frac{1}{(\pi\nu)^{\frac{n}{2}} |R|^{\frac{1}{2}}} (1+\frac{\mathbf{y}' R^{-1}\mathbf{y}}{\nu})^{-\frac{1}{2}(\nu+n)}$$

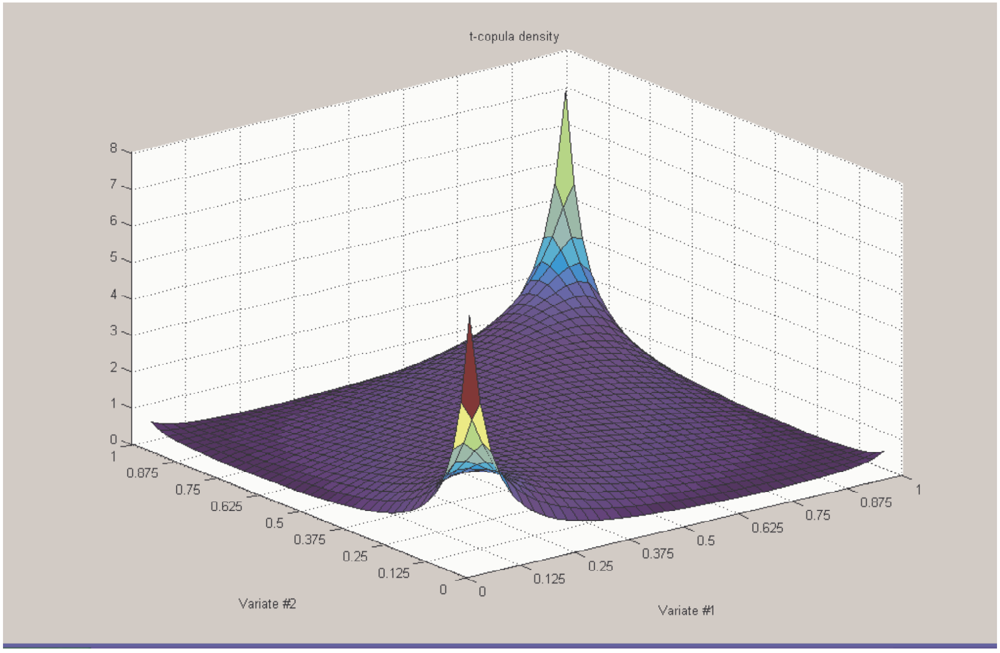

Figure 2. Student's t copula with r=0.5 and $\nu = 3$

2.2.3 Archimedean Copulae

Archimedean copulae constitute an important class of copula functions not only because of their analytical tractability (many of the most common Archimedean copulae have closed form expression) but also because they provide a large spectrum of different dependence structures.

In this section we will present three widely used copulae belonging to this class, starting from a simple bivariate framework, then extending it to a multivariate setting via the Kimberling [18] approach.

In order to analyze this class of copulae, following the analysis suggested by Nelsen [28], let us consider a function $\varphi : [0, 1] \to [0, \infty]$ such that:

- φ is continuous.
- $\varphi'(u) < 0$ for all $u \in [0, 1]$.
- $\varphi(1) = 0$.

We then define the pseudo-inverse of φ as the function $\varphi^{[-1]} : [0, \infty] \rightarrow [0, 1]$ such that:

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t) & \text{for } 0 \leq t \leq \varphi(0) \\ 0 & \text{for } \varphi(0) \leq t \leq \infty \end{cases} .$$

Now, if φ is convex, then the function $C : [0,1]^2 \rightarrow [0,1]$ defined as

$$C(u,v) = \varphi^{[-1]}\left[\varphi(u) + \varphi(v)\right] \tag{9}$$

is an Archimedean copula and φ is called the generator of the copula. Furthermore, if $\varphi(0) = \infty$, the pseudo-inverse describes an ordinary inverse function (that is $\varphi^{[-1]} = \varphi^{-1}$) and we call φ and C respectively a strict generator and a strict Archimedean copula.

Gumbel Copula Let $\varphi(t) = (-\ln t)^\theta$ with $\theta \geq 1$. Then, using equation (9) we have[8]

$$C_\theta^{Gumbel}(u,v) = \varphi^{-1}\left[\varphi(u) + \varphi(v)\right] = \exp\left\{-\left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{1/\theta}\right\}.$$

Clayton Copula Let $\varphi(t) = \left(t^{-\theta} - 1\right)/\theta$ with $\theta \in [-1, \infty) \setminus \{0\}$. Then, using (9) we have

$$C_\theta^{Clayton}(u,v) = \max\left[\left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}, 0\right].$$

Note that if $\theta > 0$, then $\varphi(0) = \infty$, and we can simplify the above expression as follows

$$C_\theta^{Clayton}(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}. \tag{10}$$

[8] In order to apply the formula depicted in (9) we should check if φ satisfies the defining properties of a generator function. This can be easily verified. Moreover, note that $\varphi(0) = \infty$, allowing us to use the equality $\varphi^{[-1]} = \varphi^{-1}$ and thus defining a strictly Archimedean copula.

Frank Copula Let $\varphi(t) = -\ln \frac{e^{-\theta t}-1}{e^{-\theta}-1}$ with $\theta \in \mathbb{R}\backslash\{0\}$. Then, using (9) we have

$$C_\theta^{Frank}(u,v) = -\frac{1}{\theta}\ln\left[1 + \frac{\left(e^{-\theta u}-1\right)\left(e^{-\theta v}-1\right)}{e^{-\theta}-1}\right].$$

We close the treatment of Archimedean copulae presenting a method to generalize our setting to a multivariate case. In doing this task we will follow the analysis suggested by Embrechts, Lindskog & McNeil [9], p. 37. For other approaches we refer the reader to Joe [15].

Theorem 6 *(Kimberling) Let $\varphi : [0,1] \to [0,\infty]$ be a continuous, strictly decreasing function such that $\varphi(0) = \infty$ and $\varphi(1) = 0$, and let φ^{-1} be the inverse of φ. Then, for all $n \geq 2$, the function $C : [0,1]^n \to [0,1]$ defined as*

$$C(u_1, u_2, \ldots, u_n) = \varphi^{-1}\left[\varphi(u_1) + \varphi(u_2) + \ldots + \varphi(u_n)\right]$$

is an n-dimensional Archimedean copula if and only if φ^{-1} is **completely monotone**[9] on $[0,\infty)$.

2.3 Tail dependence

Dependence among random variables and, more specifically, measurement of its intensity within a financial context, has recently been object of several studies and analysis (see, among others, Embrechts, Lindskog & McNeil [9], and Embrechts, McNeil & Straumann [10]). Alternative measures to the classical Pearson's linear correlation, have been introduced and extensively treated (see, for example Nelsen [28], ch. 5, for a review of different concepts and measurements of dependence and their link with the copula framework). In this section we will refer to the concept of tail dependence and its relevance in measuring the dependence between the occurrence of extreme values.

[9] A function $f(t)$ is said to be completely monotone on the interval D if it has derivatives of all order which alternate is sign, that is

$$(-1)^k \frac{d^k}{dt^k} f(t) \geq 0$$

for all t in the interior of D and $k = 0, 1, 2, 3, \ldots$

As Embrechts, Lindskog & McNeil [9] pointed out, if $\varphi^{[-1]}$ is completely monotone, then $\varphi^{[-1]}(t) > 0$ for all $t \in [0,\infty)$ and therefore, $\varphi^{[-1]} = \varphi^{-1}$.

Definition 7 *Let (X_1, X_2) be a bivariate vector of continuous random variables with marginal distribution functions F_1 and F_2. The coefficients of upper λ_U and lower λ_L tail dependence, provided that the limit $\lambda_U \in [0,1]$ (in case of upper tail dependence) and $\lambda_L \in [0,1]$ (in case of upper tail dependence) exist, are respectively given by the following expressions:*

$$\lambda_U = \lim_{u \nearrow 1} \mathbb{P}\left[X_2 > F_2^{-1}(u) | X_1 > F_1^{-1}(u)\right], \tag{11}$$

$$\lambda_L = \lim_{u \searrow 0} \mathbb{P}\left[X_2 \leq F_2^{-1}(u) | X_1 \leq F_1^{-1}(u)\right]. \tag{12}$$

If $\lambda_U \in (0,1]$ then the two random variables (X_1, X_2) are said to be asymptotically dependent in the upper tail. If $\lambda_U = 0$, then the two random variables (X_1, X_2) are said to be asymptotically independent in the upper tail. Symmetrically, if $\lambda_L \in (0,1]$ then the two random variables (X_1, X_2) are said to be asymptotically dependent in the lower tail. If $\lambda_L = 0$, then the two random variables (X_1, X_2) are said to be asymptotically independent in the lower tail.

By exploiting the relation between the bivariate distribution and survival functions[10], the definition of copula as depicted in (1) and the property described by (3), Joe [15], p. 33, provides an alternative version of equation (11):

[10] Let $F_{X_1,X_2}(x_1,x_2) = \mathbb{P}(X_1 \leq x_1, X_2 \leq x_2)$ be the bivariate distribution function for the random vector (X_1, X_2). The bivariate survival function $\widetilde{F}_{X_1,X_2}(x_1,x_2)$ is defined as follows

$$\widetilde{F}_{X_1,X_2}(x_1,x_2) = \mathbb{P}(X_1 > x_1, X_2 > x_2).$$

Now, in order to establish the relation between $F_{X_1,X_2}(x_1,x_2)$ and $\widetilde{F}_{X_1,X_2}(x_1,x_2)$, first note that the event $(X_1 > x_1, X_2 > x_2)$ can be expressed as:

$$\begin{aligned}
(X_1 &> x_1, X_2 > x_2) = \\
&= \Omega \backslash \left[(X_1 > x_1, X_2 \leq x_2) \cup (X_1 \leq x_1, X_2 > x_2) \cup (X_1 \leq x_1, X_2 \leq x_2)\right] \\
&= \Omega \backslash \left[(X_2 \leq x_2) \cup (X_1 \leq x_1, X_2 > x_2) \cup (X_1 \leq x_1, X_2 \leq x_2) \backslash (X_1 \leq x_1, X_2 \leq x_2)\right] \\
&= \Omega \backslash \left[(X_2 \leq x_2) \cup (X_1 \leq x) \backslash (X_1 \leq x_1, X_2 \leq x_2)\right],
\end{aligned}$$

and therefore, using the definition of distribution and survival function, we deduce:

$$\widetilde{F}_{X_1,X_2}(x_1,x_2) = 1 - F_{X_1}(x_1) - F_{X_2}(x_2) + F_{X_1,X_2}(x_1,x_2).$$

$$
\begin{aligned}
\lambda_U &= \lim_{u \nearrow 1} \mathbb{P}\left[X_2 > F_2^{-1}(u) | X_1 > F_1^{-1}(u)\right] \\
&= \lim_{u \nearrow 1} \frac{\mathbb{P}\left[X_2 > F_2^{-1}(u), X_1 > F_1^{-1}(u)\right]}{\mathbb{P}\left[X_1 > F_1^{-1}(u)\right]} \\
&= \lim_{u \nearrow 1} \frac{1-\mathbb{P}\left[X_1 \le F_1^{-1}(u)\right]-\mathbb{P}\left[X_2 \le F_2^{-1}(u)\right]+\mathbb{P}\left[X_2 \le F_2^{-1}(u), X_1 \le F_1^{-1}(u)\right]}{1-\mathbb{P}\left[X_1 \le F_1^{-1}(u)\right]},
\end{aligned}
$$

and therefore

$$
\lambda_U = \lim_{u \nearrow 1} \frac{[1 - 2u + C(u,u)]}{1-u}. \tag{13}
$$

A similar argument can be applied with regard to (12) leading to:

$$
\lambda_L = \lim_{u \searrow 0} \frac{C(u,u)}{u}. \tag{14}
$$

Clayton Copula Consider the Clayton copula derived in (10). In order to determine the coefficient of lower tail dependence, we apply (14) obtaining

$$
\begin{aligned}
\lambda_L^{Clayton} &= \lim_{u \searrow 0} \frac{C(u,u)}{u} \\
&= \lim_{u \searrow 0} \frac{\left(2u^{-\theta} - 1\right)^{-1/\theta}}{u} \\
&= \lim_{u \searrow 0} -\frac{1}{\theta} \left(2u^{-\theta} - 1\right)^{-\frac{1+\theta}{\theta}} \cdot \left(-2\theta u^{-(1+\theta)}\right) \\
&= \lim_{u \searrow 0} 2 \left(2u^{-\theta} - 1\right)^{-\frac{1+\theta}{\theta}} \cdot \left(u^{-\frac{(1+\theta)}{\theta}\theta}\right) \\
&= \lim_{u \searrow 0} 2 \left[u^{\theta} \left(2u^{-\theta} - 1\right)\right]^{-\frac{1+\theta}{\theta}} \\
&= \lim_{u \searrow 0} 2 \left[2 - u^{\theta}\right]^{-\frac{1+\theta}{\theta}} = 2^{-1/\theta} > 0.
\end{aligned}
$$

Therefore, if $\theta > 0$, then the Clayton copula exhibits asymptotic lower tail dependence. Furthermore, a similar calculation shows that this copula is asymptotically independent in the upper tail.

Especially when dealing with copulae without closed form expressions (such as the gaussian and the Student's t copula) an alternative version, provided by Embrechts, Lindskog & McNeil [9], p. 16, can be employed. First, by applying De L'Hopital theorem, we obtain:

$$\begin{aligned}\lambda_U &= \lim_{u\nearrow 1}\frac{[1-2u+C(u,u)]}{1-u}\\ &= -\lim_{u\nearrow 1}\left[-2+\frac{\partial}{\partial x}C(x,y)|_{x=y=u}+\frac{\partial}{\partial y}C(x,y)|_{x=y=u}\right].\end{aligned}$$

Then, by the definition of copula function and by the expression of the copula density in equation (4), we have[11]:

$$\mathbb{P}\left(V\leq v|U=u\right)=\frac{\partial}{\partial u}C(u,v),\quad \mathbb{P}\left(U\leq u|V=v\right)=\frac{\partial}{\partial v}C(u,v)$$

and

$$\mathbb{P}\left(V>v|U=u\right)=1-\frac{\partial}{\partial u}C(u,v),\quad \mathbb{P}\left(U>u|V=v\right)=1-\frac{\partial}{\partial v}C(u,v).$$

Rearranging the expression above, we obtain:

$$\lambda_U=\lim_{u\nearrow 1}\left[\mathbb{P}\left(V>u|U=u\right)+\mathbb{P}\left(U>u|V=u\right)\right]. \tag{15}$$

In the case of exchangeable copulas (i.e. $C(u,v)=C(v,u)$) we can further simplify (15) as follows

$$\lambda_U=2\lim_{u\nearrow 1}\left[\mathbb{P}\left(V>u|U=u\right)\right].$$

Finally, by using the probability-integral theorem, in the case of a distribution function F with infinite right endpoint (such as the gaussian or the Student's t distribution) we have:

$$\lambda_U=2\lim_{u\to\infty}\left[\mathbb{P}\left(F^{-1}\left(V\right)>u|F^{-1}\left(U\right)=u\right)\right]=2\lim_{u\to\infty}\left[\mathbb{P}\left(Y>u|X=u\right)\right], \tag{16}$$

where, by Sklar's theorem[12], $(X,Y)'\sim C(F(x),F(y))$.

---

[11] For a formal proof see Appendix 7.2.

[12] By the definition of copula function and by applying the probability-integral transform to a vector $(X,Y)'$ of continuous random variables with univariate margin F, we obtain:

$$\begin{aligned}F(x,y) &= \mathbb{P}(X\leq x,Y\leq y)\\ &= \mathbb{P}(F\left(X\right)\leq F\left(x\right),F\left(Y\right)\leq F(y))\\ &= \mathbb{P}(U\leq F\left(x\right),V\leq F(y))=C(F\left(x\right),F(y)).\end{aligned}$$

Gaussian Copula Consider the bivariate case of the gaussian copula with linear correlation r, described in (5):

$$C_R^{Gaussian}(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi(1-r^2)^{1/2}} \exp\left\{-\frac{x^2 - 2rxy + y^2}{2(1-r^2)}\right\} dxdy.$$

In order to apply (16), we first need to compute the conditional distribution of Y given $X = x$. To this end, since[13] $Y|X = x \sim \mathcal{N}(rx, 1-r^2)$, we have

$$\begin{aligned} \lambda_U &= 2\lim_{x\to\infty}\left[\mathbb{P}(Y > x|X = x)\right] \qquad (17) \\ &= 2\lim_{x\to\infty}\left[1 - \Phi\left(\frac{x - rx}{\sqrt{1-r^2}}\right)\right] \\ &= 2\lim_{x\to\infty}\left[1 - \Phi\left(x\frac{\sqrt{1-r}}{\sqrt{1+r}}\right)\right] = 0. \end{aligned}$$

Furthermore, given the radial symmetry property of elliptical distributions, we can conclude that also the lower tail dependence coefficient is null, confirming the asymptotic independence in tail of the gaussian copula.

Student's t copula Consider the bivariate case of the Student's t copula with linear correlation r, described in (7):

$$C_R^{Student}(u,v) = \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi(1-r^2)^{1/2}} \left[1 + \frac{x^2 - 2rxy + y^2}{\nu(1-r^2)}\right]^{-\left(\frac{\nu+2}{2}\right)} dxdy.$$

In order to apply (16) we first need to compute the conditional distribution of Y given $X = x$. With this regard it can be shown (for a formal proof refer to Appendix 7.1) that the conditional distribution $f^{Student}(Y|X = x)$ belongs to the location-scale family with standard probability density function $f^{Student}(x)$ with location parameter (mean) equal to rx and scale parameter (standard deviation) equal to $\left[\frac{(\nu+x^2)(1-r^2)}{\nu+1}\right]^{1/2}$.

We can then use this result to compute the coefficient of upper (and lower, due to the radial symmetry property) tail dependence of the Student's

[13] This is a standard result in the statistic theory. For a formal proof we refer to Casella & Berger [5], p. 177.

t copula:

$$\begin{aligned}
\lambda_U &= 2 \lim_{x\to\infty} \left[\mathbb{P}\left(Y > x | X = x\right)\right] \\
&= 2 \lim_{x\to\infty} \left\{1 - t_{\nu+1}\left[\left(\frac{\nu + x^2}{\nu+1}\right)^{-1/2} \cdot \frac{(x - rx)}{\sqrt{(1-r^2)}}\right]\right\} \\
&= 2 \lim_{x\to\infty} \left\{1 - t_{\nu+1}\left[\left(\frac{\frac{\nu}{x^2} + 1}{\nu+1}\right)^{-1/2} \cdot \frac{\sqrt{(1-r)}}{\sqrt{(1+r)}}\right]\right\} \\
&= 2 - 2t_{\nu+1}\left[\sqrt{\nu+1} \cdot \frac{\sqrt{(1-r)}}{\sqrt{(1+r)}}\right]. \qquad (18)
\end{aligned}$$

As shown in Figure 3, the coefficient λ_U, is increasing in r and decreasing in v. As the number of degrees of freedom tend to infinity, the coefficient λ_U tends to 0 for $r < 1$. This fact can be explained by considering the asymptotic behavior of a Student's t distributed random variable, which, as known[14], converges in distribution to a standard normal variate.

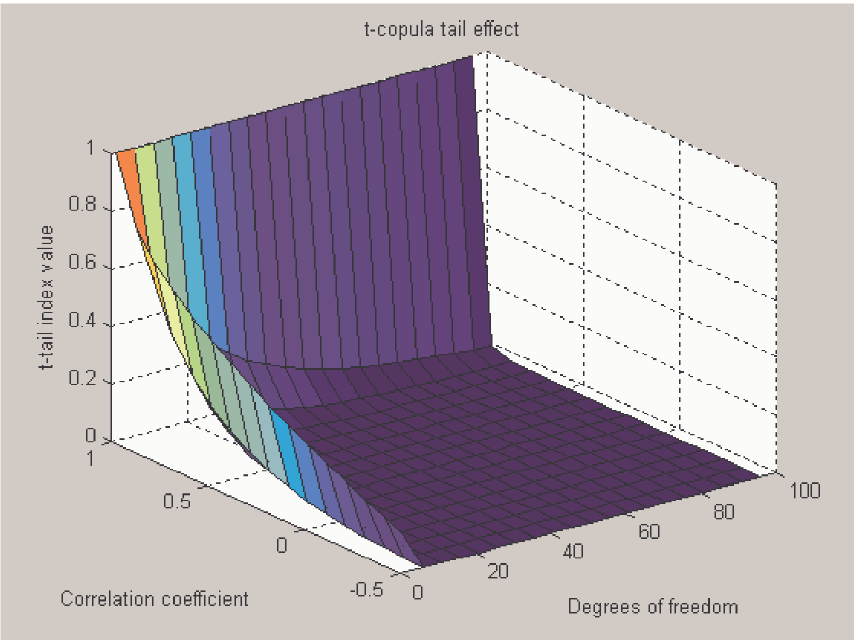

Figure 3. Value of the lower (upper) tail dependence index for the t copula

[14] For a formal proof, see Casella & Berger [5], p. 258.

2.4 Calibrating Copulae

Calibrating copula parameters to real market data represents an active research area in the current statistical literature (see, for example, Durrleman *et al.* [8], Mashal & Zeevi [25], Meneguzzo & Vecchiato [26] and Romano [30]). With this regard, in this section we will survey various approaches suggested in the literature, and we will compare them in terms of computational efficiency. Working example with real market data will be provided at the end of the section.

In the following analysis we will consider a random sample represented by the time series $X = (X_{1t}, X_{2t}, \ldots X_{Nt})_{t=1}^{T}$ where N stands for the number of underlying assets included and T represents the number of observations (on a daily, monthly, quarterly or yearly base) available.

2.4.1 Exact Maximum Likelihood Method (EML)

Let Θ be the parameter space and θ be the k-dimensional vector of parameters to be estimated. Let $L_t(\theta)$ and $l_t(\theta)$ be, respectively, the likelihhood and the log-likelihood for the observation at time t. We then define the log-likelihood function $l(\theta)$ as follows

$$l(\theta) = \sum_{t=1}^{T} l_t(\theta). \tag{19}$$

Let us now consider the canonical expression for density function as expressed by equation (4). We can then expand (19) as follows

$$l(\theta) = \sum_{t=1}^{T} \ln c\left(F_1(x_1^t), \ldots, F_N(x_N^t)\right) + \sum_{t=1}^{T} \sum_{n=1}^{N} \ln f_n\left(x_n^t\right). \tag{20}$$

We finally define the exact maximum likelihood estimator, as the vector $\widehat{\theta}$ such that

$$\widehat{\theta} := (\widehat{\theta}_1, \widehat{\theta}_2, ..., \widehat{\theta}_k) \in \arg\max \left\{l(\theta) : \theta \in \Theta\right\}.$$

Gaussian Copula Let $\Theta = \left\{R : R \in \mathbb{R}^{N\times N}\right\}$, with R being a symmetric and positive definite matrix, denote the parameter space. We can apply (20) to the case of the gaussian copula density described in equation (6), obtaining

$$l^{gaussian}(\theta) = -\frac{T}{2}\ln|R| - \frac{1}{2}\sum_{t=1}^{T}\zeta_t'\left(R^{-1} - I\right)\zeta_t. \tag{21}$$

Assuming that the log-likelihood function in (20) is differentiable in θ, and that the solution of the equation $\frac{\partial}{\partial\theta}l(\theta) = 0$ defines a global maximum we can easily recover the maximum likelihood estimator $\widehat{\theta} = \widehat{R}$ for the gaussian copula whose log-likelihood function is reported in (21):

$$\frac{\partial}{\partial R^{-1}}l^{gaussian}(\theta) = \frac{T}{2}R - \frac{1}{2}\sum_{t=1}^{T}\zeta_t'\zeta_t,$$

and therefore

$$\widehat{R} = \frac{1}{T}\sum_{t=1}^{T}\zeta_t'\zeta_t. \tag{22}$$

Student's t Copula Let $\Theta = \left\{(\nu, R) : \nu \in (2,\infty], R \in \mathbb{R}^{N\times N}\right\}$, with R being a symmetric and positive definite matrix, denote the parameter space. We can then apply (20) to the case of the Student's t copula density described in equation (8). In this case the calculation is more involved, leading to

$$\begin{aligned} l^{Student}(\theta) &= T\ln\frac{\Gamma\left(\frac{\nu+N}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} - NT\ln\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} - \frac{T}{2}\ln|R| \\ &\quad -\frac{\nu+N}{2}\sum_{t=1}^{T}\ln(1+\tfrac{\zeta_t'R^{-1}\zeta_t}{\nu}) + \frac{\nu+1}{2}\sum_{t=1}^{T}\sum_{n=1}^{N}\ln(1+\tfrac{\zeta_{nt}^2}{\nu}). \end{aligned} \tag{23}$$

Unlike the case of gaussian copula, the calibration of the Student's t copula via the EML method is more complicated since it requires a simultaneous estimation (see, for example, Johnson & Kotz [17], p. 145) of the parameters of the margins and the parameters related to the dependence structure. But this procedure, as pointed out by Mashal & Naldi [24], p. 15, needs a huge amount of data and is computationally very intensive. For that reason alternative methodologies have been introduced as reported in next paragraphs.

2.4.2 The Inference Functions for Margins method (IFM)

This method, based on the pioneering work of Joe & Xu [16], exploiting the fundamental idea of copula theory (that is, the separation between the univariate margins and the dependence structure), expresses equation (20) in the following representation

$$l(\theta) = \sum_{t=1}^{T} \ln c\left(F_1(x_1^t;\theta_1),\ldots,F_N(x_N^t;\theta_N);\boldsymbol{\alpha}\right) + \sum_{t=1}^{T}\sum_{n=1}^{N} \ln f_n\left(x_n^t;\theta_n\right). \quad (24)$$

The peculiarity of (24) relies in the separation between the vector of the parameters for the univariate marginals $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_N)$ and the vector of the copula parameters $\boldsymbol{\alpha}$. In other words, the calibration of the copula parameters to market data is performed via a two stage procedure:

1. Estimation of the vector of the parameters for the marginal univariates $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_N)$ via the EML method. For example, considering the time series of the i-th underlying asset, we have[15]

$$\widehat{\theta}_i = \arg\max_{\theta_i} \sum_{t=1}^{T} \ln f_i\left(x_i^t;\theta_i\right).$$

2. Estimation of the vector of the copula parameters $\boldsymbol{\alpha}$, using the previous estimators $\widehat{\boldsymbol{\theta}}=\left(\widehat{\theta}_1,\ldots,\widehat{\theta}_N\right)$:

$$\widehat{\boldsymbol{\alpha}}_{IFM} = \arg\max_{\boldsymbol{\alpha}} \sum_{t=1}^{T} \ln c\left(F_1(x_1^t;\widehat{\theta}_1),\ldots,F_N(x_n^t;\widehat{\theta}_N);\boldsymbol{\alpha}\right).$$

[15] Mashal & Naldi [23], p. 18, suggest a procedure for estimating the parameters of the marginals based on a numerical optimization routine for the likelihood of the univariate returns. Namely, starting from a univariate return sample, $\{X_t\}_{t=1}^{T}$ they construct a new sample $\left\{\widehat{X}_t\right\}_{t=1}^{T} = \left\{\frac{X_t-m}{H}\right\}_{t=1}^{T}$ where m is the location parameter and H is the scale factor, to be distribuited as a Student's t r.v. with ν degrees of freedom. Therefore, given a parameter space $\Theta = \{\theta : m \in \mathbb{R}, H > 0, \nu > 2\}$ with $\theta = (m, H, \nu)$, they define the Maximum Likelihood Estimator $\widehat{\theta} = \left(\widehat{m}, \widehat{H}, \widehat{\nu}\right)$ as:

$$\widehat{\theta} = \arg\max_{\theta\in\Theta} \prod_{t=1}^{T} f_{\nu}^{student}\left(\widehat{X}_t\right)$$

The IFM estimator is then defined as the vector $\boldsymbol{\theta}^{IFM} = \left(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\alpha}}_{IFM}\right)$.

Remark 2 *It is worth stressing that, in the case of the gaussian multivariate copula with gaussian margins, given that the only parameter to estimate is the correlation matrix R, the outputs of the EML and the IFM methods are equivalent.*

2.4.3 The Canonical Maximum Likelihood Method (CML)

Both the EML and IFM methods are based on an exogenous imposition of the parametric form of the univariate margins[16]. An alternative method, which does not imply any a priori assumption on the distributional form of the margins, is the CML method and relies on the concept of the *empirical marginal transformation.* This transformation tends to approximate[17] the *unknown* parametric marginals $F_n(\cdot)$, for $n = 1, \ldots, N$, with the empirical distribution functions $\widehat{F}_n(\cdot)$defined as follows

$$\widehat{F}_n(\cdot) = \frac{1}{T}\sum_{t=1}^{T} 1_{\{X_{nt} \leq \cdot\}}, \qquad for \quad n = 1, \ldots, N, \tag{25}$$

where $1_{\{X_{nt} \leq \cdot\}}$ represents the indicator function. The CML method is then implemented via a two stage procedure:

1. Transformation of the initial data set $X = (X_{1t}, X_{2t}, \ldots X_{Nt})_{t=1}^{T}$ into uniform variates, using the empirical marginal distribution, that is, for $t = 1, \ldots, T$, let $\widehat{u}_t = (\widehat{u}_1^t, \widehat{u}_2^t, \ldots, \widehat{u}_N^t) = \left[\widehat{F}_1(X_{1t}), \widehat{F}_2(X_{2t}), \ldots, \widehat{F}_N(X_{Nt})\right]$.

2. Estimation of the vector of the copula parameters $\boldsymbol{\alpha}$, via the following relation:

$$\widehat{\boldsymbol{\alpha}}_{CML} = \arg\max_{\boldsymbol{\alpha}} \sum_{t=1}^{T} \ln c\left(\widehat{u}_1^t, \widehat{u}_2^t, \ldots, \widehat{u}_N^t; \boldsymbol{\alpha}\right).$$

The CML estimator is then defined as the vector $\boldsymbol{\theta}^{CML} = \widehat{\boldsymbol{\alpha}}_{CML}$.

[16] Note that for equation (23) we have fixed $\zeta = \left(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2), \ldots, t_\nu^{-1}(u_N)\right)'$. Similarly, with regard to (21), we have $\zeta = \left(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_N)\right)'$.

[17] The Glivenko-Cantelli lemma ensures that, when sample size tends to infinity, $\widehat{F}$ converge uniformly to F on the real line, almost surely. See Mashal & Zeevi [25], p. 15, for more details.

2.4.4 Numerical Results

In this section we will present an application of the CML method for calibrating the parameter of the Student's t copula to real market data. To this end, we will consider a portfolio of four stocks (Fiat, Merrill Lynch, Ericsson and British Airways)[18] with 990 daily observations spanned from August, 14th 1999 to July, 15th 2003. Two approaches will be provided: the first one, suggested by Bouyè *et al.* [4], pp. 41-42, based on a recursive optimization procedure for the correlation matrix, the second, proposed by Mashal & Zeevi [25], p. 43, based on the rank correlation estimator given by the Kendall's Tau.

Bouye, Durrleman, Nikeghbali, Riboulet, and Roncalli approach
This procedure, composed by a series of subsequent steps, can be summarized and implemented as follows:

1. Starting from the random sample X of stock returns, transform the initial data set into the set of uniform variates $\widehat{U}$ using the empirical marginal transformation described in § 2.4.3.

2. For each value of the degrees of freedom ν on a specified range, assess the correlation matrix R_ν^{CML} using the following routine:

   i) For each fixing date t, let $\xi_t = (t_\nu^{-1}(\widehat{u}_{1t}), t_\nu^{-1}(\widehat{u}_{2t}), \ldots, t_\nu^{-1}(\widehat{u}_{Nt}))'$ for $t = 1, ..., T$.

   ii) Estimate the EML estimator $\widehat{R}$ of the correlation matrix for the gaussian copula, using equation (22). Then set $R_0 = \widehat{R}$.

[18] Bloomberg tickers:
Fiat: F IM EQUITY
Merrill Lynch: MER US EQUITY
Ericcson: ERICB SS EQUITY
British Airways: BAY LN EQUITY

iii) Obtain $R_{\nu,k+1}$ via the following recursive scheme[19]:

$$R_{\nu,k+1} = \frac{\nu+N}{T\nu}\sum_{t=1}^{T}\frac{\xi_t'\xi_t}{(1+\frac{\xi_t' R_{\nu,k}^{-1}\xi_t}{\nu})}.$$

iv) Rescale matrix entries in order to have unit diagonal elements:

$$(R_{\nu,k+1})_{i,j} = \frac{(R_{\nu,k+1})_{i,j}}{\sqrt{(R_{\nu,k+1})_{i,i}(R_{\nu,k+1})_{j,j}}}.$$

v) Repeat procedure iii)-iv) until $R_{\nu,k+1} = R_{\nu,k}$ and set $R_{\nu}^{CML} = R_{\nu,k}$.

3. Find the CML estimator ν^{CML} of the degrees of freedom by maximizing the log-likelihood function of the Student's t copula density:

$$\nu^{CML} = \arg\max_{\nu\in\Theta}\sum_{t=1}^{T}\log c^{Student}\left(\widehat{u}_1^t, \widehat{u}_2^t, \ldots, \widehat{u}_N^t; R_{\nu}^{CML}, \nu\right).$$

Figure 4 plots the log-likelihood function of the t copula density as a function of the degrees of freedom, by which we can see that the estimated number of degrees of freedom is 10.

[19] This equation is derived by the maximization output of the log-likelihood function for the t copula density reported in (23):

$$\frac{\partial l^{Student}(\theta)}{\partial R^{-1}} = \frac{T}{2}R - \frac{\nu+N}{2}\sum_{t=1}^{T}\frac{\frac{\xi_t'\xi_t}{\nu}}{(1+\frac{\xi_t' R^{-1}\xi_t}{\nu})},$$

from where we see that the ML estimator of R must satisfy the following equation:

$$R_{ML} = \frac{\nu+N}{T\nu}\sum_{t=1}^{T}\frac{\xi_t'\xi_t}{(1+\frac{\xi_t' R_{ML}^{-1}\xi_t}{\nu})}.$$

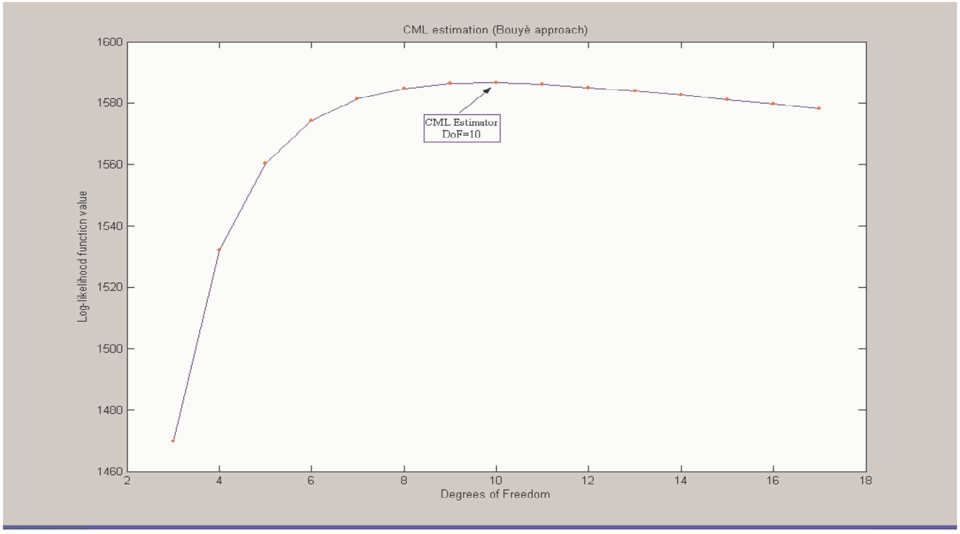

Figure 4. CML estimator for the degrees of freedom of the t copula (Bouyè approach)

The corresponding correlation matrix R_{10}^{CML} is presented in Table (26):

1	0.34771	0.81475	0.77631
0.34771	1	0.62666	0.65706
0.81475	0.62666	1	0.77288
0.77631	0.65706	0.77288	1

Correlation Matrix (26)

Mashal and Zeevi method The procedure described above can be computationally heavy when dealing with several underlying assets and large data sets, and (as pointed out by Mashal & Zeevi [25], p. 41) affected by numerical instability due to the inversion of close to singular matrices. For this reason, the two authors propose to estimate the correlation matrix for the Student's t copula via a rank correlation estimator, namely the Kendall's tau[20], exploiting the result included in the following theorem:

Theorem 8 *Let $X \sim E_N(\mu, \Sigma, \varphi)$, where for $i, j \in \{1, 2, ..., N\}$, X_i and X_j are continuous. Then,*

$$\tau(X_i, X_j) = \frac{2}{\pi} \arcsin R_{ij}, \tag{27}$$

where $E_N(\mu, \Sigma, \varphi)$ denotes the N-dimensional elliptical distribution with parameters (μ, Σ, φ), and $\tau(X_i, X_j)$ and R_{ij} indicate, respectively, the Kendall's tau and the Pearson's linear correlation coefficient for the random variables (X_i, X_j).
Proof. *: see Lindskog, McNeil & Schmock [22], p. 5.*

With this result at hand, the methodology proceeds as follows:

[20] We adopt the definition provided by Embrechts, Lindskog & McNeil [9]. Let (x, y) and $(\widetilde{x}, \widetilde{y})$ be two observations from a vector (X, Y) of continuous random variables. Then (x, y) and $(\widetilde{x}, \widetilde{y})$, are said to be concordant if

$$(x - \widetilde{x})(y - \widetilde{y}) > 0$$

and discordant otherwise. Kendall's tau is then defined as the probability of concordance minus the probability of discordance, namely, given two pairs (X, Y) and $(\widetilde{X}, \widetilde{Y})$ of independent random variables with the same distribution $F(.,.)$, we have:

$$\tau = \mathbb{P}\left[(X - \widetilde{X})(Y - \widetilde{Y}) > 0\right] - \mathbb{P}\left[(X - \widetilde{X})(Y - \widetilde{Y}) < 0\right]$$

As suggested by Meneguzzo & Vecchiato [26], p. 56, given two series X_t and Y_t with $t = 1, ..., T$, the consistent estimator of Kendall's tau is then computed as follows

$$\tau = \frac{2}{T(T-1)} \sum_{i<j} sgn\left[(X_i - X_j)(Y_i - Y_j)\right],$$

where $sgn(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$.

1. Starting from the random sample X of stock prices, transform the initial data set into the set of uniform variate $\widehat{U}$ using the empirical marginal transformation described in § 2.4.3.

2. From equation (27) estimate the correlation matrix R^{CML}.

3. Find the CML estimator ν^{CML} of the degrees of freedom by maximizing the log-likelihood function of the Student's t copula density:

$$\nu^{CML} = \arg\max_{\nu\in\Theta} \sum_{t=1}^{T} \log c^{Student}\left(\widehat{u}_1^t, \widehat{u}_2^t, \ldots, \widehat{u}_N^t; R^{CML}, \nu\right).$$

Figure 5 plots the log-likelihood function of the t copula density as a function of the degrees of freedom, by which we can see that the estimated number of degrees of freedom is 9.

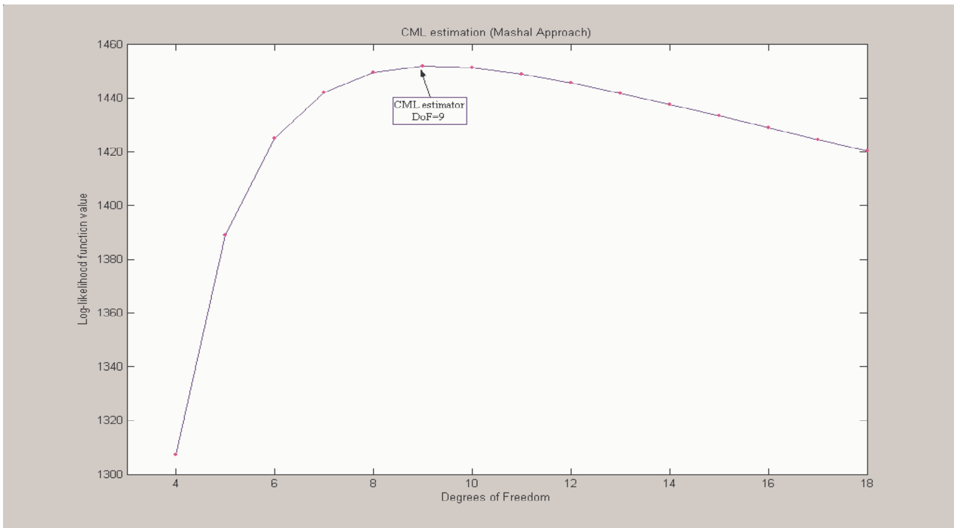

Figure 5. CML estimator for the degrees of freedom of the t copula (Mashal approach)

The corresponding correlation matrix R^{CML} is presented in Table (28)

1	0.44818	0.90208	0.83975
0.44818	1	0.67615	0.68552
0.90208	0.67615	1	0.84178
0.83975	0.68552	0.84178	1

Correlation matrix (28)

3 Constructing a Credit Curve

A fundamental part of the pricing framework we are going to present in order to model multiname credit derivative products is represented by the estimation of the instantaneous default probabilities for each reference entity. With this regard, several methods has been proposed in the literature (for a complete exposition, see, for example, Bluhm *et al.* [2], p. 183) which can be summarized as follows:

- From rating agencies' databases of historical default rates.
- Using the Merton (1974) model and its extensions (see, Delianedis & Geske [6]).
- Using market observable data (asset swap spreads, defaultable bond prices, credit default premia) to estimate the implicit default term structure.

In our analysis we will focus on the third approach showing how to construct a term structure of instantaneous default probabilities (commonly referred as credit curve) starting from the information derived from credit default swap premia. To this end we will introduce the concept of hazard function and present some standard results on Poisson and Cox stochastic processes, highlighting the relation between the hazard rate and the intensity of the Poisson (Cox) process so as to aid the comprehension of the Monte Carlo approach to pricing basket default swaps and CDOs which is presented in the following chapters. § 3.4 will terminate with a practical implementation of the model with real market data.

3.1 Hazard Rate Function

Given a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$, let τ be the $(\mathcal{F}_t)$-stopping time at which default occurs, with a distribution function $F(t) = \mathbb{P}(\tau \leq t)$ and probability density function $f(t)$. Then, indicating with $\mathbb{P}(t < \tau \leq t + \Delta t | \tau > t)$ the probability that default will occur in the interval $(t,\, t + \Delta t)$ given that the reference entity has survived up to time t, we define the hazard function $h(t)$ as follows:

$$h(t) = \lim_{\Delta t \to 0} \mathbb{P}(t < \tau \leq t + \Delta t | \tau > t).$$

Expanding the expression above, it is possible to recover an interesting relation between $h(t)$, $f(t)$ and $F(t)$:

$$\begin{aligned} h(t) &= \lim_{\Delta t \to 0} \frac{\mathbb{P}(t < \tau \le t + \Delta t, \tau > t)}{\mathbb{P}(\tau > t)} = \lim_{\Delta t \to 0} \frac{\int_t^{t+\Delta t} f(u)du}{\int_t^{\infty} f(u)du} \\ &= \frac{f(t)}{1 - F(t)} = \frac{\frac{\partial}{\partial t} F(t)}{1 - F(t)} = -\frac{\partial}{\partial t} \log\left(1 - F(t)\right), \end{aligned}$$

and, by solving the differential equation, we have

$$F(t) = 1 - \exp\left(-\int_0^t h(u)du\right) \tag{29}$$

and

$$f(t) = h(t) \exp\left(-\int_0^t h(u)du\right). \tag{30}$$

3.2 Poisson and Cox Processes

Let $N(t)$ be the non-decreasing, integer valued process defined on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, \mathbb{P}\right)$ such that:

$$N(t) := \begin{cases} \sum\limits_{i \in \mathbb{N}} 1_{\{\tau_i \le t\}} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases},$$

where $1_{\{\tau_i \le t\}}$ is the indicator function of the $(\mathcal{F}_t)$-stopping time τ_i. Then the process $N(t)$ is called (inhomogeneous) Poisson process with intensity $\lambda(t)$[21], if, for $0 \le t < T$, the following two conditions are satisfied[22]:

- The increment $N(T) - N(t)$ is independent of the σ-algebra $\mathcal{F}_t$.
- The increment $N(T) - N(t)$ is distributed according to a Poisson law with parameter $\Lambda(T) - \Lambda(t) = \int_t^T \lambda(u)du$:

$$\mathbb{P}\left[N_T - N_t = n\right] = \frac{1}{n!} \left(\int_t^T \lambda(u)du\right)^n \exp\left(-\int_t^T \lambda(u)du\right). \tag{31}$$

[21] We assume that the function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is non-negative, locally integrable and $\int_0^{\infty} \lambda(u)du = \infty$.

[22] For an intuitive construction of an inhomogeneous Poisson process adapted to a default risk framework, see Schonbucher [33], pp. 112-116.

Now, if we define the time of default as the time of the first jump of the process $N(t)$, that is

$$\tau = \inf \{t \in \mathbb{R}_+ | N(t) > 0\}, \tag{32}$$

we can establish an important relation with the analysis presented in § 3.1.

From (29), if we set $S(t) = 1 - F(t) = \mathbb{P}(\tau > t)$, we have $S(t) = \exp\left(-\int_0^t h(u)du\right)$. Similarly, in the framework of default arrival time described by (32) we have $S(t) = \mathbb{P}[N(t) = 0|\mathcal{F}_0]$, which, by equation (31), can be written as:

$$S(t) = \exp\left(-\int_0^t \lambda(u)du\right). \tag{33}$$

and hence $h(t) = \lambda(t)$.

If we extend our setting allowing the intensity λ to be a non-negative stochastic process, then $N(t)$ defines a Cox process with intensity $\lambda(t)$. More precisely, as pointed out by Lando [19], $N(t)$ is a Cox process if, conditioning on a particular realization of $\lambda(\cdot, \omega)$, $N(t)$ describes an inhomogeneous Poisson process with intensity $\lambda(t, \omega)$. The corresponding jump probabilities can be recovered using the tower property of conditional expectation, yielding to:

$$\begin{aligned}
\mathbb{P}[N_T - N_t = n] &= \mathbb{E}\left[1_{\{N_T - N_t = n\}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[1_{\{N_T - N_t = n\}}|\lambda\right]\right] \\
&= \mathbb{E}\left[\frac{1}{n!}\left(\int_t^T \lambda(u)du\right)^n \exp\left(-\int_t^T \lambda(u)du\right)\right],
\end{aligned}$$

and therefore

$$S(t) = \mathbb{E}\left[\exp\left(-\int_0^t \lambda(u)du\right)\right] \tag{34}$$

and

$$f(t) = \mathbb{E}\left[\lambda(t)\exp\left(-\int_0^t \lambda(u)du\right)\right].$$

3.3 Calibration of the hazard rate function

As reported before, one of the most widely adopted method for estimating the instantaneous default probabilities, consists in inferring them from market observable data (see Li [20] for an application with defaultable bond prices). The basic idea is to derive the implied default probabilities from the credit default swap premia of different maturities, and then use these risk neutral probabilities to value more exotic structures (such as the multiname products to be described in the next sections). To this end we consider a probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}^*\right)$, with $\mathbb{P}^*$ being the equivalent martingale measure under which price processes of any tradeable security, discounted by the money market account, are $\mathbb{P}^*$-martingales with respect to the filtration $(\mathcal{F}_t)$. Typically, three fundamental assumptions are postulated:

1. The interest rate process and the default process are independent under the martingale measure $\mathbb{P}^*$.
2. Defaults can only happen at a set of finite, discrete dates.
3. Default payments are settled immediately upon default.

Under these assumptions, we define the risk neutral price of a credit default swap (CDS) with maturity T, as the fraction k of the nominal value M (expressed in basis points per annum) such that the default leg (contingent payment in case of default) and the premium leg (stream of fixed cash flow to be payed at time $(t_1, t_2, ..., t_n = T)$ either until T or until $\tau < T$ in case of default) are equal in present value. Formally, we define the premium leg PL as the expected value, under the probability measure $\mathbb{P}^*$, of the series of cash flows indexed to k, discounted at risk free zero coupon rate, to be payed until maturity (or earlier in case of default before T):

$$
\begin{aligned}
PL &= M \sum_{i=1}^{n} \mathbb{E}^* \left[k\Delta_i B(0,t_i) 1_{(\tau > t_i)} \right] \qquad (35) \\
&= M \sum_{i=1}^{n} k\Delta_i B(0,t_i) \left[1 - F(t_i) \right] \\
&= M \sum_{i=1}^{n} k\Delta_i B(0,t_i) \exp \left[-\int_0^{t_i} h(u) du \right],
\end{aligned}
$$

where Δ_i denotes the day count fraction on the interval (t_{i-1}, t_i) and $B(0, t_i)$ is the deterministic risk free discount factor for the interval $(0, t_i)$.

Similarly, we define the default leg DL as the expected value, under the probability measure $\mathbb{P}^*$, of the present value of the contingent payment (function of the recovery rate R, assumed to be deterministic) upon default DP, minus the contingent accrued premium AP[23]:

$$DL = DP - AP,$$

where

$$\begin{aligned} DP &= M\mathbb{E}^*\left[(1-R)\,B(0,\tau)1_{(\tau\leq T)}\right] \\ &= M\,(1-R)\int_0^T B(0,u)F(du) \\ &= M\,(1-R)\int_0^T B(0,u)h(u)\exp\left(-\int_0^u h(s)ds\right)du \end{aligned}$$

and

$$\begin{aligned} AP &= Mk\sum_{i=1}^n \mathbb{E}^*\left[\frac{\tau - t_{i-1}}{t_i - t_{i-1}}\Delta_i B\,(0,\tau)\,1_{\{t_{i-1}<\tau\leq t_i\}}\right] \\ &= Mk\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{u - t_{i-1}}{t_i - t_{i-1}}\Delta_i B(0,u)F(du) \\ &= Mk\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{u - t_{i-1}}{t_i - t_{i-1}}\Delta_i B(0,u)h(u)\exp\left(-\int_0^u h(s)ds\right)du. \end{aligned}$$

The fair price (spread) k^* is subsequently defined as follows:

$$k^* \Rightarrow PL(k^*) - DL(k^*) = 0,$$

[23] If t_i is the premium payment date prior to the default date τ, then the accrued premium AP to be payed by the protection buyer, is computed as:

$$AP = Nk\Delta_{i+1}\frac{\tau - t_i}{t_{i+1} - t_i}.$$

and therefore we obtain

$$k^* = \frac{(1-R)\int_0^T B(0,u)F(du)}{\sum_{i=1}^n \Delta_i B(0,t_i)\exp\left[-\int_0^{t_i} h(u)du\right] + \int_{t_{i-1}}^{t_i} \frac{u-t_{i-1}}{t_i-t_{i-1}}\Delta_i B(0,u)F(du)}. \quad (36)$$

From (36), we see that the fair price k^* reflects both the market perception of the loss severity given default (through the factor R) and the default probabilities (through the hazard function $h(t)$). Therefore, assuming some exogenously (from rating agencies databases, based on the asset's seniority) specified recovery rate R, it is possible to extract the risk neutral default probabilities from the market spreads k of CDSs with different maturities.

3.4 Numerical Results

In this section we will provide a practical application of model described in § 3.3 for extracting the term structure of instantaneous default probabilities from default swap quotes. In our analysis[24] we are going to consider the same portfolio of securities used in § 2.4.4.

In the following table we report the market quotes for credit default swaps of the four reference entities examined:

Issuer/Maturity	$1y$	$2y$	$3y$	$4y$	$5y$
Fiat Spa	750/850	740/840	720/820	670/740	630/680
Ericsson	275/375	335/415	450/500	435/485	470/480
British Airways Plc	450/550	450/550	450/550	450/550	400/500
Merrill Lynch Inc.	30/34	30/35	33/37	35/39	39/43

CDS Quotes (bid/ask), semiannually compounded, in basis points per annum (as at July, 17th 2003)

A common assumption in the calibration methodology, postulates that hazard rates are piecewise constant between the CDS maturity dates. If we denotes with $[T_1, T_2, ..., T_M]$ the expiry (in years) of the CDS contracts available in the market, this implies that the hazard function $h(t)$ can be expressed as follows

$$h(t) = \sum_{i=1}^{M} \alpha_i 1_{\{T_{i-1},T_i\}}(t),$$

[24] The discount factors $B(0,t_i)$ are derived by applying the bootstrapping procedure to the Euribor and the Euro Swap curve as at August, 17th 2003.

for some positive constants α_i and $i = 1, ..., M$.

This assumption implies that the distribution function $F(t)$ in (29) can be written as

$$F(t) = 1 - \exp\left[-\sum_{j=1}^{k} \alpha_j \left(T_j - T_{j-1}\right)\right], \quad k = \begin{cases} 1 & \text{if } t \leq t_1 \\ 2 & \text{if } t_1 < t \leq t_2 \\ ... & ... \\ M & \text{if } t > t_{M-1} \end{cases} \tag{37}$$

By inserting (37) into the pricing equation (36) for the CDS with the shortest maturity T_1, and appropriately fixing the recovery rate for each reference entity, we can recover α_1. Knowing α_1 we can calibrate α_2 using the market spread relative to the CDS expiring at T_2, and in this fashion calibrate all the remaining α_j's up to time T_M.

With regard to the issuers considered, Figure 6 shows the results of the calibration procedure for different assumptions on recovery rates.

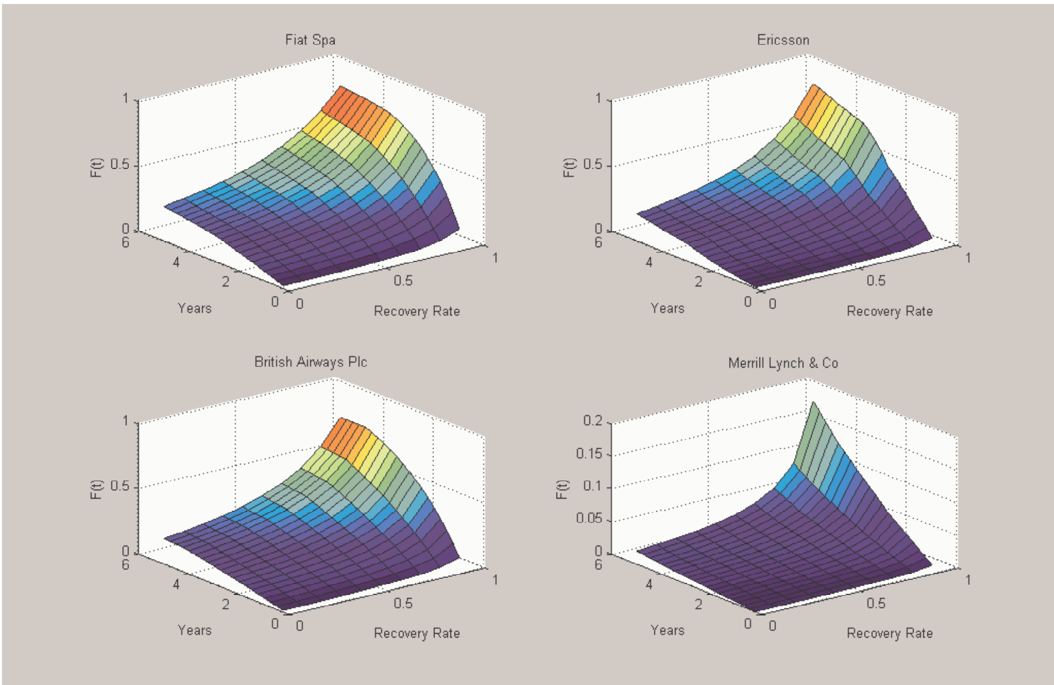

Figure 6. Default time distribution functions extracted from market quotes

4 Basket Default Swaps Pricing

In this section we are going to consider the problem of multiname credit derivatives pricing within the copula framework developed in previous sections. This part is organized as follows: § 4.1 will present a generalized simulation algorithm for generating correlated defaulted times, whose consistency with initial market data will be guaranteed by calibrating our model to the initial individual term structure of instantaneous default probabilities computed as described in § 3.3. A comparison between the gaussian and the Student's t copula will be provided and the role of tail dependence highlighted in order to explain the different default dynamics derived by these two different dependency structures. In § 4.2 we will study the valuation of k-th to default basket swaps, starting from the original setting introduced by Li [21] and Schmidt & Ward [32]. § 4.4 will analyze the influence of changes of different price drivers (correlation, number of reference entities, single name CDS spreads) on the mark-to-market position of basket default swaps for different assumptions on seniority (first, second and third to default) and dependence structure (gaussian and Student's t copulae).

4.1 Generation of Correlated Default Stopping Times

The simulation of default times within a portfolio context is intimately related with the specific choice of the copula function used to describe the dependency structure among obligors and to the parametric form of the univariate margins used to define the survival functions for each entity. Based on this observation Li [21] proposed to decompose the simulation algorithm into a two stage procedure:

1. Generation of correlated uniform random variables on $[0, 1]$ from an N-dimensional copula.

2. Translation into default times via the inverse of the marginal distribution function.

While the first step is specific to the particular choice of copula function, the second is connected with the default time arrival setting discussed in § (3.2).

Therefore, our analysis will proceed as follows. In § 4.1.1 we will present the techniques used for sampling random variables from elliptical copulae. More precisely, following the approach developed in Embrechts, Lindskog & McNeil [9], we will provide the simulation algorithms for sampling with the gaussian and the Student's t copula. § 4.1.2 will then discuss the general Monte Carlo algorithm for estimating the distribution of the default stopping times for the obligors included in a basket structure.

4.1.1 Sampling from Elliptical Copulae

Multivariate Gaussian Copula The procedure for generating random variables from the gaussian copula $C_R^{Gaussian}$ with correlation matrix R proceeds as follows:

1. Find a suitable (e.g. Cholesky) decomposition A of R, such that $R = A \cdot A^T$.
2. Draw an N-dimensional vector $\mathbf{z} = (z_1, z_2, ..., z_N)'$ of uncorrelated standard normal variates.
3. Set $\mathbf{x} := \mathbf{z}'A$.
4. Set $\mathbf{x}$ back to an N-dimensional vector $\mathbf{u}$ of uniform variates on $[0, 1]$ by computing $\mathbf{u} = \Phi(\mathbf{x})$.

Then, $\mathbf{u} \sim C_R^{Gaussian}$.

Multivariate Student's t Copula The procedure for generating random variables from the gaussian copula $C_{R,\nu}^{Student}$ with correlation matrix R and ν degrees of freedom proceeds as follows:

1. Find a suitable (e.g. Cholesky) decomposition A of R, such that $R = A \cdot A^T$.
2. Draw an N-dimensional vector $\mathbf{z} = (z_1, z_2, ..., z_N)'$ of uncorrelated standard normal variates.
3. Draw an independent χ^2_ν random variable s.
4. Set $\mathbf{y} := \mathbf{z}'A$.

5. Set $\mathbf{x} := \mathbf{y}\sqrt{\frac{\nu}{s}}$.

6. Map $\mathbf{x}$ back to an N-dimensional vector $\mathbf{u}$ of uniform variates on $[0, 1]$ by computing $\mathbf{u} = t_\nu(\mathbf{x})$.

Then[25], $\mathbf{u} \sim C^{Student}_{R,\nu}$.

Figure 7 shows 3000 samples from the gaussian and t copula; it is interesting to note how, compared with the gaussian case, samples from the t copula accumulate in the upper right and lower left region, thus confirming the impact of the tail dependence in modeling the occurrence of extreme events.

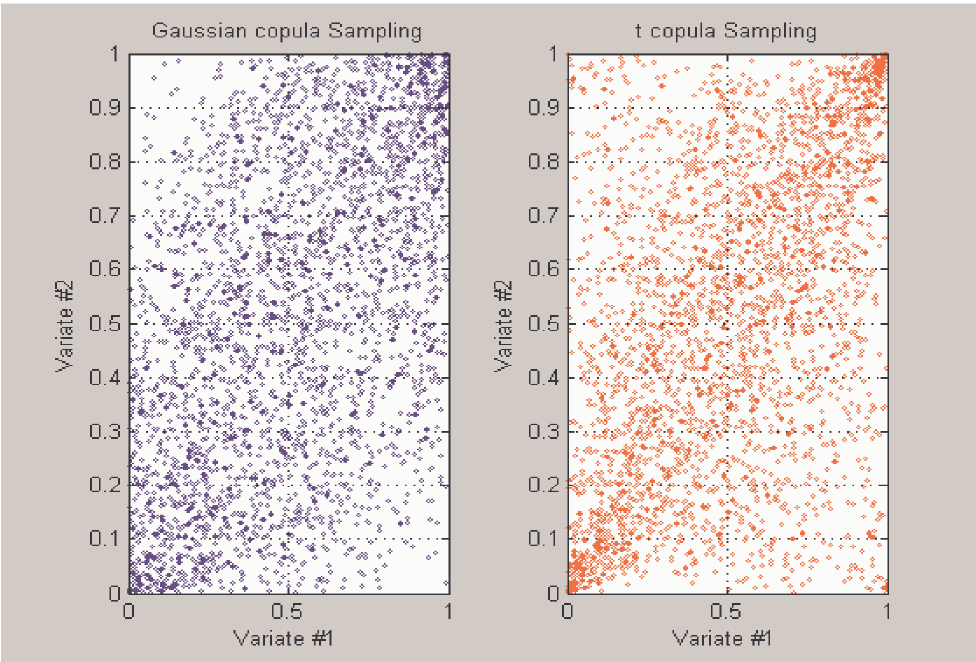

Figure 7. Sampling with the gaussian (left) and t copula (right) with $r = 0.5$, $\nu = 2$ and 3000 runs.

[25]The algorithm can be easily implemented in Matlab; let R be the correlation matrix, N the number of credits and DoF the degrees of freedom. Then we have:

```
A=chol(R);
z=randn(N,1);
s=chi2rnd(DoF);
y=z'*A;
x=(sqrt(DoF)/sqrt(s))*y;
u=tcdf(x,DoF);
```

4.1.2 The distribution of default arrival times

We consider the framework introduced in § 3.2, by letting, as suggested by Lando [19], the stochastic process $\lambda(t)$ having the form $\lambda(t) = \lambda(X_t)$, where the state variable X is an $\mathbb{R}^d$-valued stochastic process and $\lambda : \mathbb{R}^d \to [0, \infty)$ is a non-negative, continuous function. Relative to the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\right)$, we denote with $(\mathcal{G}_t)_{t \geq 0}$ the filtration generated by X and we assume that all the default-free processes (spot rates, stock prices, etc.) and the intensity process $\lambda(t)$ are $(\mathcal{G}_t)$-adapted. Moreover we let $(\mathcal{H}_t)_{t \geq 0}$ be the filtration generated by the counting process $N(t)$, which, for every $t \geq 0$, reveals the information about defaults up to time t. The full filtration $(\mathcal{F}_t)_{t \geq 0}$ is then defined as $(\mathcal{F}_t)_{t \geq 0} = (\mathcal{G}_t)_{t \geq 0} \vee (\mathcal{H}_t)_{t \geq 0}$, where $\mathcal{G}_t = \sigma \{X_s : 0 \leq s \leq t\}$ and $\mathcal{H}_t = \sigma \left\{1_{\{\tau \leq s\}} : 0 \leq s \leq t\right\}$, and hence represents the information set, on the path of the state variable X and defaults occurred, available at time t. As mentioned before, conditioning on $\mathcal{G}$ (which, by definition, contains knowledge of the realization of λ), $N(t)$ describes an inhomogeneous Poisson process with intensity $\lambda(t)$ and the distribution function and density of the first jump of $N(t)$ are given respectively by (29) and (30). Based on this property, Schonbucher [33], p. 215, observes that, for every simulation run, the path of the default intensity is known and, therefore, suggests the following simulation procedure in order to estimate the distribution of default for the N obligors:

1. Simulate an N-dimensional vector $\mathbf{u} = (u_1, u_2, ..., u_N)'$ of uniform variates[26] from a copula C.

2. Calibrate the intensities $\lambda_n(t)$ of survival function in equation (33) using the methodology exposed in § 3.4 for each reference entity $n \in \{1, 2, ..., N\}$.

3. For each obligor define the process $\gamma_n(t) := \ln S_n(t) = -\int_0^t \lambda_n(s)ds$.

4. For each obligor, compute the simulated arrival time of default as follows
$$\tau_n := \inf \{t > 0 : \gamma_n(t) \leq \ln u_n\}. \tag{38}$$

[26] A fundamental assumption in the setup suggested by Lando [19], postulates that the vector $\mathbf{u}$ is independent of $\mathcal{G}$.

Remark 3 *In our setting, we assume that the intensities $\lambda_n(t)$ are piecewise constant between the maturity dates $[T_1, T_2, ..., T_M]$ of the credit default swap contracts used in the calibration procedure. This implies that $\lambda_n(t) = \sum_{i=1}^{M} \alpha_i^n 1_{\{T_{i-1},T_i\}}(t)$ and therefore the process $\gamma_n(t)$ can be computed as follows:*

$$\gamma_n(t) := \ln S_n(t) = -\sum_{j=1}^{k} \alpha_j^n \left(T_j - T_{j-1}\right), \quad k = \begin{cases} 1 & \text{if } t \leq t_1 \\ 2 & \text{if } t_1 < t \leq t_2 \\ ... & ... \\ M & \text{if } t > t_{M-1} \end{cases}$$

A very useful result due to Schonbucher [33], pp. 216-17, proves indeed that the above algorithm replicates the correct distribution of the default stopping times. To this end, we know from (34) that

$$\mathbb{P}(\tau_n \geq t) = \mathbb{E}\left[\exp\left(-\int_0^t \lambda_n(s)ds\right)\right] = \mathbb{E}\left[\exp\left(\gamma_n(t)\right)\right].$$

If we define $\widetilde{\tau}_n$ the time at which $\gamma_n(\widetilde{\tau}_n) = \ln u_n$, then, by (38), we have

$$\mathbb{P}(\widetilde{\tau}_n \geq t) = \mathbb{P}(\gamma_n(t) \geq \ln u_n).$$

Since, by assumption, $\mathbf{u}$ is independent from $\mathcal{G}_t$ for all $t > 0$, by the tower property of conditional expectation we have

$$\begin{aligned} \mathbb{P}\left(\widetilde{\tau}_n \geq t\right) &= \mathbb{E}\left[\mathbb{P}(\ln u_n \leq \gamma_n(t) | \gamma_n(t))\right] \\ &= \mathbb{E}\left\{\mathbb{P}\left[u_n \leq \exp\left(\gamma_n(t)\right)\right]\right\} \\ &= \mathbb{E}\left[\exp\left(\gamma_n(t)\right)\right] \\ &= \mathbb{P}(\tau_n \geq t), \end{aligned}$$

proving that the simulated time $\widetilde{\tau}_n$ and the default time τ_n (time of the first jump of the Cox process $N_n(t)$ with intensity λ_n) have the same distribution.

4.2 Pricing of k-th to default basket swap

We first set the notation used throughout the analysis:

- N is the number of reference entities.
- M is the notional amount of the contract, which in the homogeneous case, coincides with the nominal amount of each reference obligation in the basket.
- $T = t_n$ is the legal maturity of the contract, measured in years from the current time $t_0 = 0$.
- k defines the seniority level of the structure; that is, a default payment is due by the protection seller upon the k-th default at time $\tau_{(k)}$ with $\tau_{(1)} < ... < \tau_{(k)} < ... < \tau_{(N)}$.[27]
- s is the fair price of the contract, expressed as a fraction (in basis points per annum) of M, to be payed $1/\Delta$ times[28] per year either until T or until $\tau_{(k)} < T$ in case of default.
- AP is the accrued premium to be payed from the last payment date before the credit event, until the time $\tau_{(k)}$ of the k-th default.
- $B(0,t)$ is the risk free discount factor (assumed to be a deterministic function of time).
- $R_{(k)}$ is the recovery rate for the k-th defaulter (assumed to be deterministic upon default).

Under the same assumptions given in § 3.3, the risk neutral price of the k-th to default basket swap is computed by equating the expected value of the discounted premium payment leg with the expected value of the discounted default leg, under the equivalent martingale measure $\mathbb{P}^*$.

[27] We assume that there are no joint defaults at exactly the same time.

[28] Assuming a 30/360 day count convention, we have $\Delta = 1$ for payments with annual frequency, $\Delta = 1/2$ for payments with semiannual frequency and $\Delta = 1/4$ for payments with quaterly frequency.

Formally, if we denote with $F_{(k)}(t) = \mathbb{P}^*(\tau_{(k)} \leq t)$ the distribution function of $\tau_{(k)}$, we define the premium leg PL of a k-th to default basket swap as follows

$$
\begin{aligned}
PL &= \mathbb{E}^*\left[\sum_{i=1}^{n}(sM\Delta)\,B\,(0,t_i)\,1_{\left\{\tau_{(k)}>t_i\right\}}\right] \\
&= \sum_{i=1}^{n}(sM\Delta)\,B\,(0,t_i)\left[1-F_{(k)}(t_i)\right].
\end{aligned}
\tag{39}
$$

Similarly[29] the default leg DL can be expressed as the difference between the expected discounted default payment DP and the expected discounted accrued premium AP:

$$DL = DP - AP,$$

where

$$
\begin{aligned}
DP &= \mathbb{E}^*\left[M\sum_{j=1}^{N}\left(1-R_{(k)}\right)B\left(0,\tau_{(k)}\right)1_{\left\{\tau_{(k)}\leq T\right\}}\right] \\
&= M\sum_{j=1}^{N}\left(1-R_{(j)}\right)\int_0^T B\,(0,t)\,F_{(k)}^{k^{th}=j}(dt)
\end{aligned}
\tag{40}
$$

and

$$
\begin{aligned}
AP &= \mathbb{E}^*\left[\sum_{i=1}^{n}M\left(s\frac{\tau_{(k)}-t_{i-1}}{t_i-t_{i-1}}\Delta\right)B\left(0,\tau_{(k)}\right)1_{\left\{t_{i-1}<\tau_{(k)}\leq t_i\right\}}\right] \\
&= sM\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{u-t_{i-1}}{t_i-t_{i-1}}\Delta B\,(0,u)\,F_{(k)}(du).
\end{aligned}
\tag{41}
$$

Hence, the fair price of the k-th to default basket swap, is the spread s^* such that:

$$s^* \Rightarrow PL\,(s^*) - DL(s^*) = 0,$$

[29] We indicate with $F_{(k)}^{k^{th}=j}(t)$ the distribution function of the k-th order statistic of default times relative to the j-th defaulter, that is, $F_{(k)}^{k^{th}=j}(du) = \mathbb{P}^*(\tau_{(k)} \in du$ and k-th defaulter $= j$-th obligor). This feature of the model is essential for allowing different recovery assumptions for each reference entity in the basket.

and we deduce

$$s^* = \frac{\sum_{j=1}^{N} \left(1 - R_{(j)}\right) \int_0^T B\left(0, t\right) F_{(k)}^{k^{th}=j}(dt)}{\sum_{i=1}^{n} \Delta B\left(0, t_i\right) \left[1 - F_{(k)}(t_i)\right] + \int_{t_{i-1}}^{t_i} \frac{u - t_{i-1}}{t_i - t_{i-1}} \Delta B\left(0, u\right) F_{(k)}(du)}. \tag{42}$$

4.3 Numerical Results

In the current analysis, we are going to consider the same portfolio of reference entities considered in §§ 2.4.4 and 3.4. The pricing methodology then proceeds as follows:

1. For each reference entity calibrate the parameters of the default time distribution function by computing the implied default intensities $\lambda_n(t)$ for $t = [T_1, T_2, ..., T_M]$ being the set of expiry dates of credit default swaps available in the market [see § 3.4].

2. Calibrate the parameters of the copula function chosen for modeling the dependence among the obligors in the basket [see § 2.4.4].

3. For each simulation repeat the following routine:

   (a) Generate an N-dimensional vector of correlated uniform random variables using the algorithms presented in § 4.1.1.

   (b) For each obligors, translate the corresponding uniform variate into default time using the procedure presented in § 4.1.2.

   (c) Sort the N-dimensional vector of default time τ in ascending order, and, according to the seniority of the contract (k-th to default), select the k-th coordinate.

   (d) Based on the specific realization of the order statistic $\tau_{(k)}$, compute the discounted value of the premium payments, accrued interest and default payment.

4. Compute the arithmetic averages of the above quantities and apply equation (42) to determine the fair spread s^*.

Adopting the Student's t copula calibration procedure developed in § 2.4.4 and thus fixing the correlation matrix equal to the one in Table (28) and letting the degrees of freedom ν equal to 9, we calculated the theoretical value of a 5 years maturity first to default ($k = 1$) basket swap, obtaining the following results[30]:

1^{st} to default premium	
Gaussian copula	818.84
Student's t copula, $\nu = 9$	816.13
(bps per annum, annually compounded)	

As clear from the above table, results form the two elliptical copulae are close to each other. In the next paragraph we will explain this fact by showing the impact of the tail dependence relative to different assumptions on correlation, number of reference entities in the basket and seniority.

4.4 Sensitivity Analysis

Default basket swap prices are logically exposed to several parameters such as correlation between obligors, size and credit worthiness of the pool of reference entities, recovery values and seniority. In this section we will study the relative influence of each of these factors in the risk/return profile of basket structures under the following set of assumptions:

- The underlying credit default swap curve is flat and constant among different reference entities.
- The risk free zero coupon curve is flat at 5%.
- The recovery is deterministic and equal to 50% for all the obligors.
- The correlation among obligors is pairwise constant.
- The premium is payed annually, with a "30/360" day count convention.
- The number of simulation runs is set at 200,000.

[30]We ran 500,000 simulation adopting the "antithetic variates" variance reduction technique.

4.4.1 Correlation Exposure

Figure 8 shows the premium of a 5 years expiry first to default basket with 5 and 20 names with constant credit default swap premia equal to 50 bps. As shown, the first to default premium is monotonically decreasing in correlation, irrespectively of the copula function (normal or Student's t) used to model dependency across obligors. A simple no-arbitrage argument can help in explaining this property: if the correlation among obligors in null and the term structure of CDS premia is flat, then it is possible to perfectly hedge a short position on a first to default contract by holding a long position on each of the credit default swaps on the reference entities in the basket. This is true because, upon default, the payment required on the first to default will be compensated by the positive cash flow on the corresponding single CDS; moreover, given the absence of correlation, the default of one obligor will not affect the credit spreads of the remaining credits thus enabling the hedger to unwind the corresponding CDSs at zero cost. This proves that, in case of zero correlation, the first to default premium has to be approximately equal to the sum of the underlying CDS premia. Conversely, the higher the correlation, the higher the probability of multiple defaults and, hence, lower the degree of protection (and thus the premium required) offered by a first to default contract compared to the strategy of holding each of the underlying CDSs.

Furthermore, the premium computed within the gaussian framework always dominates the corresponding premium with the Student's t copula. This fact can be explained using the results obtained in § 2.3. Student's t copula is characterized by fatter tails, and according to the tail dependence index presented in equation (18), allows for more extreme joint credit events to happen, thus increasing the default correlation among obligors and subsequently lowering the fair price requested to buy/sell protection. The impact of this difference is, nevertheless, dependent on the correlation among the credits in the portfolio: by (17) we argued that for the gaussian copula joint extreme events occur almost independently from each other; but high values of correlation, even in the gaussian framework, increase the probability of joint extreme events, thus reducing the difference in premium with respect to the Student's t copula.

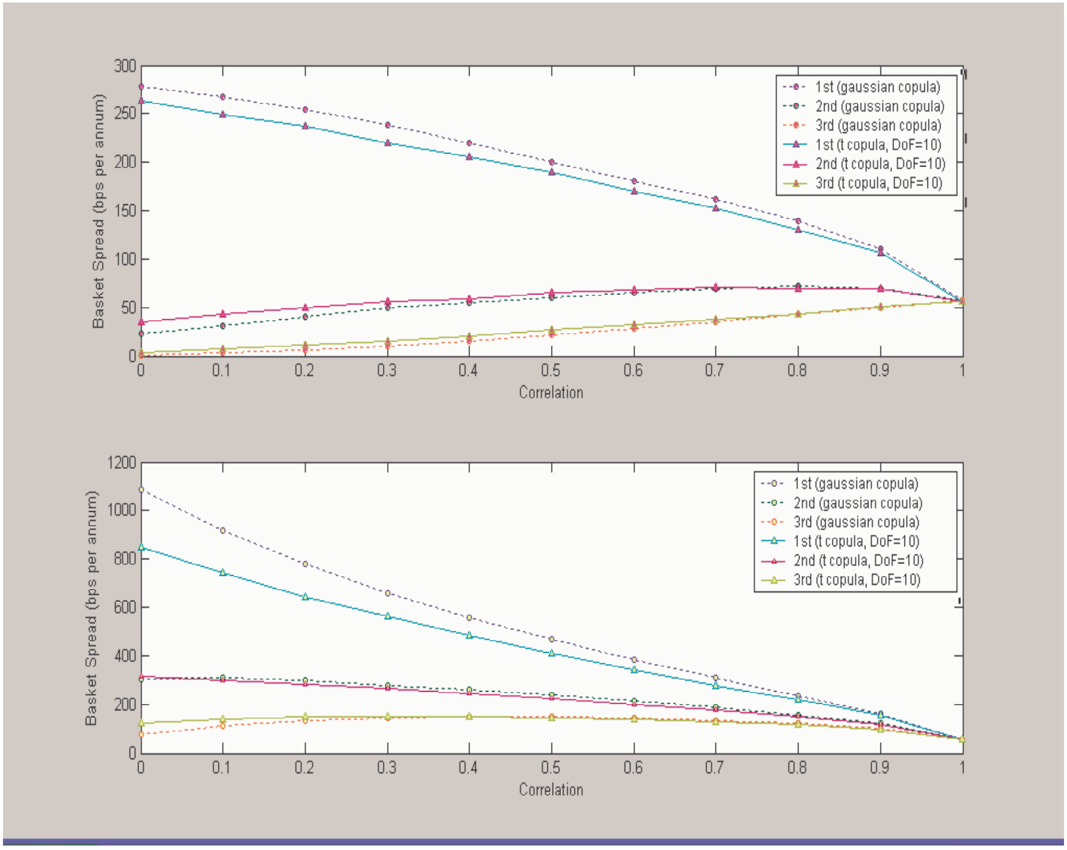

Figure 8. Basket default swap prices for $N = 5$ (above) and $N = 20$ (below)

The effect on correlation on basket default prices becomes less predictable for other orders of seniority. Even though, for small number of reference credits, second to default premium generally increases with correlation, this relation cannot be claimed for baskets referencing larger number of obligors; Naldi [27] motivates this different sensitivity by arguing that, when linear correlation varies, two effects come into play:

i) variation in the correlation among defaulters needed to trigger the default payment;

ii) variation in the correlation between the remaining credits.

While i) increases the probability of joint credit events and thus the premium of the contract, ii) affects second to default swaps in the same way as for first to defaults, lowering the requested premium. In a portfolio with a relatively small number of reference entities, i) dominates over ii) making second to default premia monotonically increasing over a large portion of correlation values. Relatively to the specific dependence structure (gaussian or t copula), an argument similar to the one provided for first to default baskets can be applied; the asymptotic tail dependence of the t copula clearly affects prices, especially for low values of correlation; as correlation increases, joint extreme events are more likely to happen in both cases, making price differences relatively small. The opposite happens with large portfolios; i) tends to be offset by ii) making second to default swap premia decreasing.

Remark 4 *In a first to default contract, i) does not affect prices, since only one defaulter is needed to trigger the contingent payment. Therefore the shape of the curve remains monotonically decreasing with respect to correlation, irrespectively of the number of credits referenced by the contract.*

4.4.2 Recovery Rate

Figure 9 exhibits how basket default swap prices vary according to different assumptions on the estimated recovery rate on the defaulted obligor:

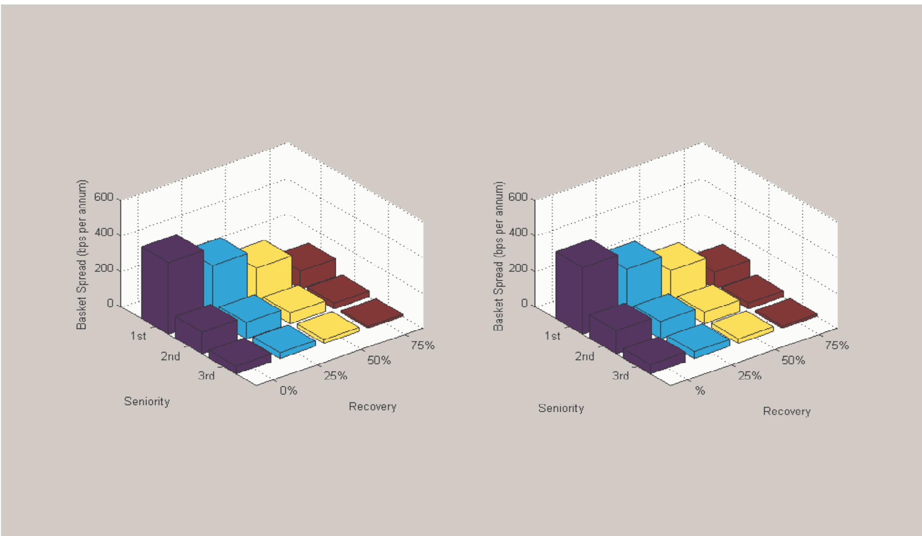

Figure 9. Basket default swap prices with gaussian (left) and Student's t_{10} (right) copula for $r = 0.5$

Higher recovery rates decrease the expected loss on the collateral portfolio, thus lowering the required premium on the contract for all orders of seniority.

4.4.3 Credit worthiness of the reference pool

Let us suppose to have a 5 names basket default swap with the same set of assumptions as above (constant recovery, correlation and credit default swap premia). We now want to study the influence derived by shifts of the underlying CDS curve on prices of first and second to default swaps. At first glance, it might be reasonable to guess that the mark-to-market sensitivity of second to default swaps, being less exposed to default risk, should be less volatile than first to default baskets. However we can show that, under certain assumptions, this is not verified in practice, especially for baskets with relatively few reference entities. To this end, the analysis developed in the previous section can help us to motivate this claim.

Figure 10 shows prices for first and second to default basket for different values of correlation and underlying CDS spread curve:

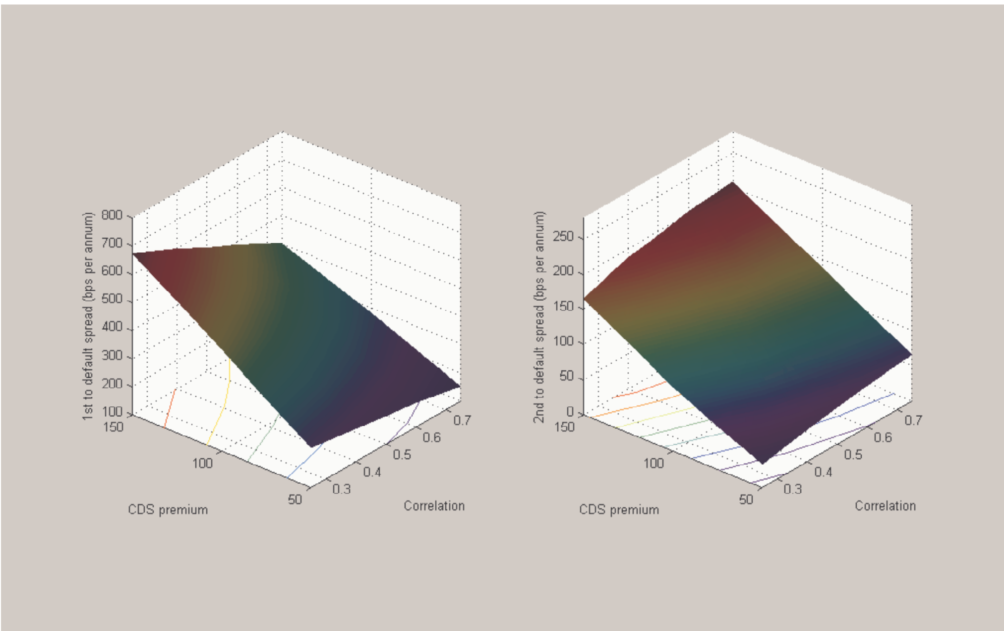

Figure 10. First (left) and second (right) to default prices surface computed with gaussian copula

Coherently with the results of § 4.4.1 we see that first to default spreads are monotonically decreasing with respect to correlation, and the opposite is true for second to default baskets. Moreover, as expected, both the structures show a positive dependence with respect to the credit default swap premia of the underlying credits. But what is really relevant to our analysis, is that the effect on prices, due to changes in either correlation and CDS premia, is different according to the seniority of the structure. Second to default baskets, even if less exposed to default risk, exhibit a higher mark-to-market volatility, as shown by the more pronounced inclination of the prices surface on the correlation-CDS premium plane.

To this end, we have calculated price changes of a first and second to default basket starting from an initial state of constant pairwise correlation $r_{ij} = r$ of 50% and flat spread curve set at 50 basis points:

	1^{st} to default				2^{nd} to default
Spread	$r=$ **0.6**	$r=$ **0.7**	**Spread**	$r=$ **0.6**	$r=$ **0.7**
75*bps*	31.52%	7.831%	75*bps*	78.80%	93.50%
100*bps*	70.51%	39.97%	100*bps*	148.80%	165.01%
125*bps*	108.08%	70.24%	125*bps*	221.45%	229.08%
150*bps*	143.19%	100.44%	150*bps*	291.42%	300.76%

Price changes computed with gaussian copula

As announced, in terms of exposure to market risk, changes in correlation and single name CDS spreads affect more heavily second to default baskets rather than first to defaults. Based on the results obtained in § 4.4.1 this phenomenon can be easily explained: we have seen that first to default spreads are monotonically decreasing in correlation, while the opposite is true for second to default baskets with a relatively small number of credits. Thus, the positive mark-to-market due to the higher CDS premia, for first to default contracts, is partially compensated by the loss due to the higher correlation, whereas, for second to default baskets, is amplified by the higher correlation, inducing a greater overall volatility.

5 Collateralized Debt Obligations Pricing

The copula framework developed in previous sections for valuing defaultable claims can be successfully extended and applied even in the case where the size of the reference pool of obligors considerably increases. A typical example is provided by an important asset class, the Collateralized Debt Obligations (CDOs), where the underlying portfolio of credits can, on average, reference more than 100 obligors. For additional information on market structure, product diversification and financial innovation in CDOs we refer to Isla [14] and O'Kane [29]. Our analysis is organized as follows: after a specification of the loss distribution for different CDO tranches in § 5.1, §§ 5.2 and 5.3 will present a copula based pricing methodology for pricing CDO tranches in the spirit of the procedure implemented for default basket swaps. In § 5.4 we will study the impact of different assumptions about price drivers (correlation and recovery rates) on the fair prices of CDO tranches. § 5.5 will finally analyze how different dependency structures (e.g. gaussian and Student's t copulae) affect the pricing results, by highlighting the role of the asymptotic tail dependence in modeling the expected losses of CDOs.

5.1 Loss Distribution

We first set the notation used throughout the analysis:

- N is the number of reference entities included in the collateral pool.
- A_i is the nominal amount of the i-th reference obligation.
- R_i is the deterministic recovery rate of the i-th reference obligation.
- $T = t_n$ is the legal maturity of the contract, measured in years from the current time $t_0 = 0$.
- τ_i is the default stopping time for the i-th obligor, which, in our context, is the time of the first jump of a Cox Process N with intensity $\lambda_i(t)$ [see § 3.2].
- $B(0, t)$ is the risk free discount (assumed to be deterministic).
- s is the fair price of the CDO tranche, expressed in basis points per annum, as a fraction of the outstanding tranche notional.

We denote with $L_i = (1 - R_i) A_i$ and $Q_i(t) = 1_{\{\tau_i < t\}}$ respectively the loss given default and the default indicator at time t the for the i-th obligor. We then define the accumulated loss $L(t)$ on the collateral portfolio as

$$L(t) = \sum_{i=1}^{N} L_i Q_i(t). \tag{43}$$

The distribution of losses between note holders will vary according to the seniority of the tranches; if we let Γ and Δ be the lower and upper triggers, the default leg of a tranche will be given by the contingent stream of payments, due to the losses occurred in the collateral pool, above the threshold Γ and below Δ. The seniority of the tranche is then defined by the relative location of the two threshold s Γ and Δ. If $\Gamma = 0$, then we speak of the *equity* tranche; if $\Gamma > 0$ and $\Delta < \sum_{i=1}^{N} A_i$, we consider the *mezzanine* tranche, and if $\Delta = \sum_{i=1}^{N} A_i$ we call the tranche *senior.*

Hence, the cumulative loss $L^{\Gamma,\Delta}(t)$ on a given tranche, will be zero if $L(t) < \Gamma$, equal to $L(t) - \Gamma$ if $\Gamma \leq L(t) < \Delta$, and $\Delta - \Gamma$ if $L(t) \geq \Delta$. Formally, we have

$$L^{\Gamma,\Delta}(t) = [L(t) - \Gamma]\, 1_{\{\Gamma,\Delta\}}(L(t)) + (\Delta - \Gamma)\, 1_{\left\{\Delta,\sum_{i=1}^{N} A_i\right\}}(L(t)). \tag{44}$$

5.2 Pricing equation

Similarly to the analysis presented for basket default swaps, the fair price of a CDO tranche is defined by the equivalence of the default (DL) and premium (PL) legs.

With regard to the former, using the representation given by Gregory & Laurent [12], we can express it as the expected value of the default payments stream, discounted from the time of default:

$$DL = \mathbb{E}^* \left[\int_0^T B(0,t) dL^{\Gamma,\Delta}(t) \right]. \tag{45}$$

The premium leg can be written as the expectation of the present value of premium payments, weighted by the outstanding capital (original amount minus accumulated losses) at each payment date, to be payed $1/\alpha$ times[31]

[31] Assuming a 30/360 day count convention, we have $\alpha = 1$ for payments with annual frequency, $\alpha = 1/2$ for payments with semiannual frequency and $\alpha = 1/4$ for payments with quaterly frequency.

per year:

$$PL = \mathbb{E}^* \left[\sum_{i=1}^{n} s_{\Gamma,\Delta} \alpha B(0, t_i) \min \{ \max [\Delta - L(t_i), 0], \Delta - \Gamma \} \right]. \tag{46}$$

In equation (46) we note that, in case of no defaults in the collateral pool (or, up to a number of defaults such that the accumulated losses are less than Γ), the discounted premium is weighted to the total notional amount of the tranche; in case of losses between Γ and Δ, the reference nominal amount is accordingly reduced, until being equal to 0 when the cumulative losses exceed the upper threshold Δ.

The fair price of the CDO tranche is then defined to be the spread $s^*_{\Gamma,\Delta}$ such that

$$s^*_{\Gamma,\Delta} \Rightarrow PL\left(s^*_{\Gamma,\Delta}\right) - DL(s^*_{\Gamma,\Delta}) = 0,$$

and hence

$$s^*_{\Gamma,\Delta} = \frac{\mathbb{E}^* \left[\int_0^T B(0,t) dL^{\Gamma,\Delta}(t) \right]}{\mathbb{E}^* \left[\sum_{i=1}^{n} \alpha B(0, t_i) \min \{ \max [\Delta - L(t_i), 0], \Delta - \Gamma \} \right]}. \tag{47}$$

Despite the compactness of the above formula, the computational effort needed to determine the distribution of the accumulated losses and, therefore, the spread s^*, is quite involved as the following section will show.

5.3 Simulation algorithm

The algorithm presented here, consists in a Monte Carlo procedure specifically developed for estimating the loss distribution of the collateral pool of a CDO and its reflection in computing the fair spread $s^*_{\Gamma,\Delta}$ of a tranche with lower and upper loss threshold s given respectively by Γ and Δ. The first part of the procedure closely resembles the algorithm presented for default basket swaps; furthermore, given the high-dimensionality connotation of the problem (collateral pools with at least 50 reference entities), a variance reduction technique is highly recommended in order to lower the number of simulations required for convergence. The procedure can be summarized as follows:

1. For each reference entity calibrate the parameters of the default time distribution function by computing the implied default intensities $\lambda_n(t)$ for $t = [T_1, T_2, ..., T_M]$ being the set of expiry dates of credit default swaps available in the market [see § 3.4].

2. Calibrate the parameters of the copula function chosen for modeling the dependence among the obligors in the pool [see § 2.4.4].

3. For each simulation k repeat the following routine:

   (a) Generate an N-dimensional vector of correlated uniform random variables using the algorithms presented in § 4.1.1.

   (b) For each obligors, translate the corresponding uniform variate into default time using the procedure presented in § 4.1.2.

   (c) Sort the N-dimensional vector of default time τ^k in ascending order, and select the vector of default times $\Upsilon^k = \left(\tau_1^k, \tau_2^k, ..., \tau_L^k\right)$ such that $\tau_j^k \leq T \quad \forall j \in \{1, 2, ..., N\}$.

   (d) Calculate the stream of contingent default payments according to the following routine:

      i. Based on the specific realization of the vector Υ^k, compute the accumulated loss in the collateral pool by calculating $L^k(T)$ as described in equation (43).

      ii. If $L^k(T)$ is less than Γ, the default payments are set equal to 0.

      iii. If $\Gamma \leq L^k(T) < \Delta$, select the default trigger time $\tau_\gamma^k = \inf\{t > 0 | L(t) \geq \Gamma\}$ and for each defaulter $l \in \{1, 2, ..., L\}$ whose default time is greater or equal to τ_γ^k compute the obligor's discounted default payment DP_w^k, namely, given the collection of default times $\Upsilon_{\Gamma,\Delta}^k = \left(\tau_\gamma^k, \tau_{\gamma+1}^k, ..., \tau_L^k\right)$, compute[32] $DP_w^k = B(0, \tau_w^k) L_w = B(0, \tau_w^k)(1 - R_w) A_w$ for $w \in$

[32] Let us consider the lower default trigger time τ_γ^k and its corresponding defaulter. We stress that only the defaulter's loss exceeding the threshold Γ has to be taken into account. That means that, for example, if the lower threshold is $\Gamma = 3\%$, the relative loss of the γ-th defaulter is $L_\gamma = 1\%$ and the accumulated loss is $L(\tau_\gamma^k) = 3.5\%$, then only the exceeding part $L(\tau_\gamma^k) - \Gamma = 0.5\%$ has to be included in the calculation of DP_w^k for the first $w = \gamma$. For the remaining defaulters, the condition $\Gamma \leq L^k(T) < \Delta$, ensures that the accumulated losses are, by assumption, below the upper threshold, thus allowing us to include in the calculation procedure the full corresponding losses.

$\{\gamma, \gamma+1, ..., L\}$ and, finally, sum over all the specified defaulters, that is $DP^k = \sum_{w=\gamma}^{L} DP_w^k$.

iv. If $L^k(T) \geq \Delta$ select τ_γ^k as described above, and calculate the upper default trigger time $\tau_\delta^k = \inf\{t > 0 | L(t) \geq \Delta\}$. Then, for each defaulter $l \in \{1, 2, ..., L\}$ whose default time is greater or equal to τ_γ^k and less than τ_δ^k, compute the discounted default payment DP_w^k, namely, given the collection of default times $\Upsilon_{\Gamma,\Delta}^k = \left(\tau_\gamma^k, \tau_{\gamma+1}^k, ..., \tau_\delta^k\right)$, compute[33] $DP_w^k = B(0, \tau_w^k) L_w = B(0, \tau_w^k)(1 - R_w) A_w$ for $w \in \{\gamma, \gamma+1, ..., \delta\}$ and, finally, sum over all the specified defaulters, that is $DP^k = \sum_{w=\gamma}^{\delta} DP_w^k$.

(e) Calculate the premium payments leg according to the following routine:

i. Based on the specific realization of the vector Υ^k, for each one of the premium payment dates $(t_1, t_2, .., t_n)$ compute the accumulated loss $L^k(t_i)$ for $i \in \{1, 2, ..., n\}$ and then calculate the premium leg

$$PL^k = \sum_{i=1}^{n} \delta B(0, t_i) \min\left\{\max\left[\Delta - L^k(t_i), 0\right], \Delta - \Gamma\right\}.$$

4. Calculate the arithmetic average of DP^k and PL^k and apply equation (47) to determine the fair spread $s_{\Gamma,\Delta}^*$.

[33] The same considerations presented in the previous note have to be included for $w = \gamma$. Moreover a similar argument has to be applied with regard to the upper default trigger time τ_δ^k and its corresponding defaulter. We stress that only the defaulter's loss **not** exceeding the threshold Δ has to be taken into account for the last $w = \delta$. That means that, for example, if the upper threshold is $\Delta = 15\%$, the relative loss of the δ-th defaulter is $L_\delta = 2\%$ and the accumulated loss is $L(\tau_\delta^k) = 16\%$, then the portion of loss to be discounted is given by $L_\delta - \left(L(\tau_\delta^k) - \Delta\right) = 1\%$. For the previous defaulters (with the eventual correction for $w = \gamma$), the condition $\Gamma \leq L^k(T)$, ensures that the accumulated losses are grater than the lower threshold, thus allowing us to include in the calculation procedure the full corresponding losses.

5.4 Numerical Results

In the present context we are going to provide an application of the above mentioned algorithm by considering a stylized CDO structure composed by an homogeneous collateral pool ($N = 100$ reference entities with the same notional amount $A_i = 100$) with a constant single name default swap curve set at $S = 150$ basis points p.a. and constant pairwise correlation $r_{ij} = r$. The remaining characteristics of the CDO considered are reported in the following table:

Reference CDO	
Maturity	*5y*
Tranche Equity (absorbed loss)	*0%-3%*
Tranche Mezzanine (absorbed loss)	*3%-14%*
Tranche Senior (absorbed loss)	*14%-100%*
Zero Coupon Curve	*flat at 5% p.a.*

The figures reported below, show the price surfaces of the three tranches, according to different assumption on recovery rates (assumed constant for all the reference entities) and correlation:

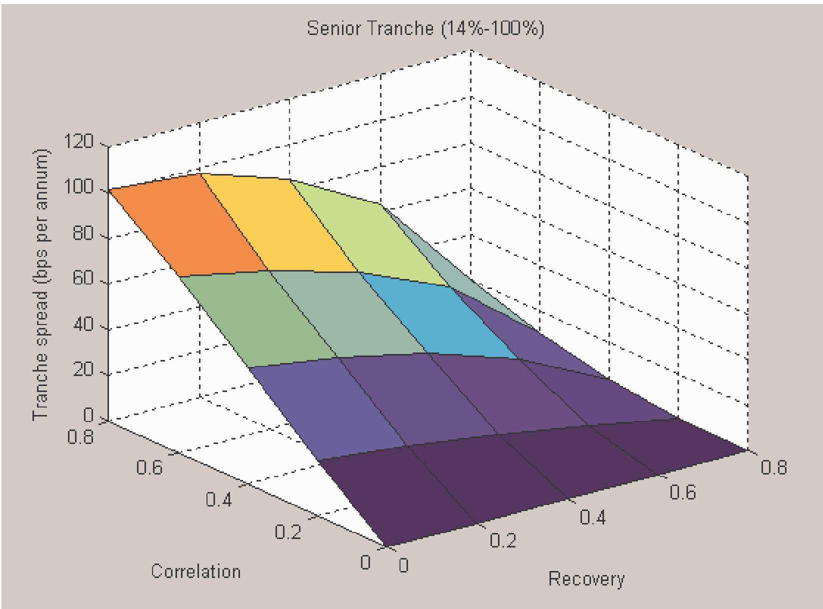

Figure 11. Senior CDO tranche priced with gaussian copula with 200,000 simulation runs

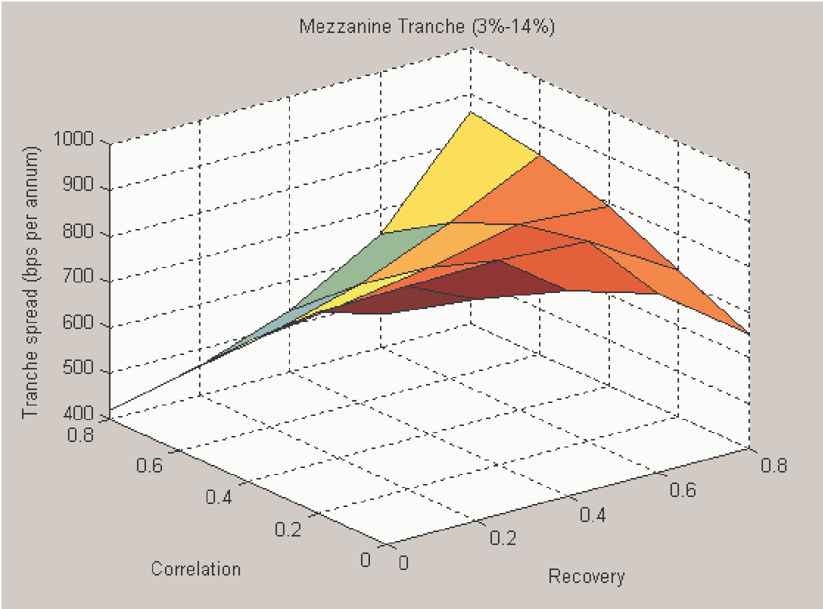

Figure 12. Mezzanine CDO tranche priced with gaussian copula with 200,000 simulation runs

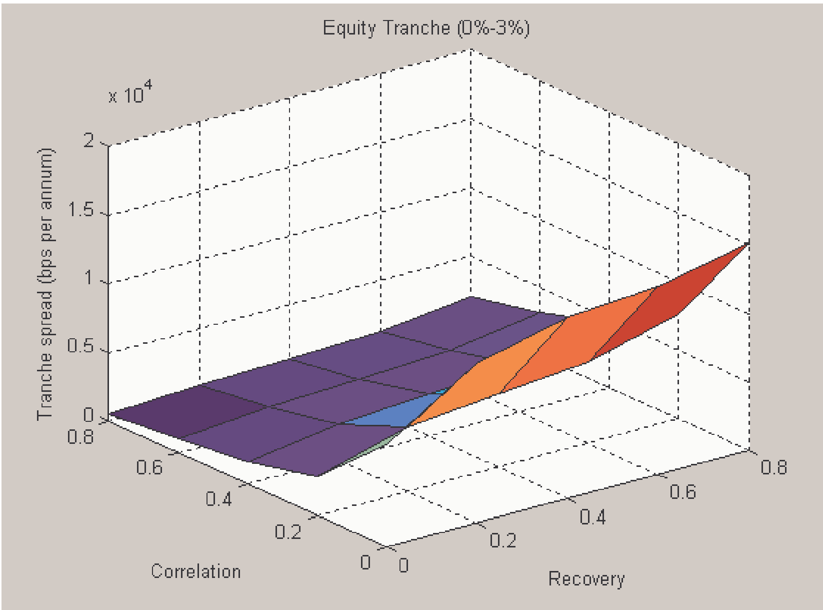

Figure 13. Equity CDO tranche priced with gaussian copula with 200,000 simulation runs

As a general result, it can be highlighted that there are different relationships between prices and correlation for CDO tranches: senior tranches show a positive relation between prices and correlation, while the opposite is verified for the equity piece. Mezzanine tranches seem to behave similarly to the equity piece as far as correlation sensitivity is concerned, especially for recovery values under 60%; the dependence on recovery values is nevertheless less straightforward, since, for higher correlation values, the sign of the relation resembles the behavior of the junior tranche.

With regard to the recovery dependence, while senior tranche spreads appear to be monotonically decreasing, the opposite is verified for equity tranches. This behavior, first described in Boscher & Ward [3], p. 128, even if counterintuitive at first glance, can be reasonably explained by analyzing the shape of loss distribution of the collateral pool for different assumptions on the recovery rate. To this end, starting from (43) we can compute the first two moments of the loss distribution, obtaining:

$$\begin{aligned}\mathbb{E}\left[L(t)\right] &= \mathbb{E}\left[\sum_{i=1}^{N} L_i Q_i(t)\right] \\ &= \sum_{i=1}^{N} L_i \mathbb{E}\left[1_{\{\tau_i<t\}}\right] \\ &= \sum_{i=1}^{N} \left(1-R_i\right) A_i F_i(t), \end{aligned} \tag{48}$$

$$\mathbb{V}\left[L(t)\right] = \sum_{i=1}^{N} \mathbb{V}\left[L_i(t)\right] + \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} r_{ij} \sqrt{\mathbb{V}\left[L_i(t)\right] \mathbb{V}\left[L_j(t)\right]}, \tag{49}$$

where $\mathbb{V}\left[L_i(t)\right]$ stands for the single obligor loss variance, defined by

$$\mathbb{V}\left[L_i(t)\right] = \left(1-R_i\right)^2 A_i^2 F_i(t)\left[1-F_i(t)\right].$$

In the case of homogeneous collateral pool (same notional A for each reference entity and same recovery rate R), with constant pairwise correlation $r_{ij}=r$ and flat credit default swap curve[34], we have $\mathbb{V}\left[L_i(t)\right]=\mathbb{V}\left[L_j(t)\right]$ for all $i,j\in\{1,2,...,N\}$; this implies that (49) can be rewritten as follows

$$\mathbb{V}\left[L(t)\right] = N\mathbb{V}\left[L_i(t)\right] + N(N-1)r\mathbb{V}\left[L_i(t)\right].$$

[34]The assumption of flat spread curve implies an interesting relation between the con-

We then define the unexpected loss UL as the square root of the portfolio loss variance

$$\begin{aligned} UL(t) &= \sqrt{N\mathbb{V}\left[L_i(t)\right] + N(N-1)r\mathbb{V}\left[L_i(t)\right]} \\ &= \sqrt{N + N(N-1)r}\,(1-R)\,A\sqrt{F(t)\,(1-F(t))}. \end{aligned} \tag{50}$$

It is clear that the dispersion of the loss distribution, denoted by UL, is an increasing function with respect to the correlation r. Regarding the effect of the recovery rates, by setting $\sqrt{N + N(N-1)r}A = Q$ and differentiating (50) with respect to the recovery rate R we obtain

$$\begin{aligned} \frac{\partial UL(t)}{\partial R} &= Q\left[\frac{\partial\,(1-R)}{\partial R}\sqrt{F(t)\,(1-F(t))} + (1-R)\,\frac{\partial\sqrt{F(t)\,(1-F(t))}}{\partial R}\right] \\ &= Q\left[-\sqrt{F(t)\,(1-F(t))} + \frac{St\,(1-R)^{-1}\,(1-F(t))\,(1-2F(t))}{2\sqrt{F(t)\,(1-F(t))}}\right]. \end{aligned}$$

Fixing $W = \frac{St}{(1-R)}$, after simple algebraic manipulations, we have

$$\frac{\partial UL(t)}{\partial R} = Q\sqrt{1-F(t)}\left[-\sqrt{F(t)} + W\frac{1-2F(t)}{2\sqrt{F(t)}}\right].$$

For $R < 1$ and $S > 0$, the term in square brackets is negative, thus showing that the unexpected loss (the standard deviation) of the collateral pool is a monotonically decreasing function of the recovery rate.

---

tract spread, the hazard rate and the recovery rate, namely

$$h = \frac{S}{1-R}.$$

For a formal proof we refer to Meneguzzo & Vecchiato [26], pp. 54-55. Furthermore, based on this property, it is possible to express the default time distribution function in (37) as follows:

$$\begin{aligned} F\,(t) &= 1 - \exp\left(-ht\right) \\ &= 1 - \exp\left(-\frac{S}{1-R}t\right). \end{aligned}$$

Figure 14 plots the loss distribution of the CDO collateral pool ($N = 100$) as a function of different values of the recovery rate:

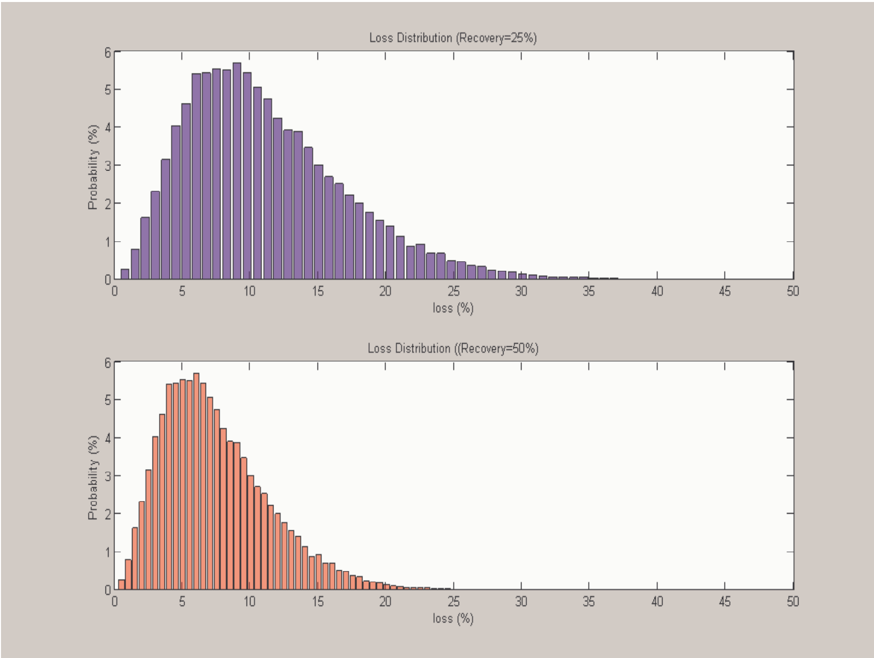

Figure 14. Collateral pool simulated losses ($r = 0.1$ and constant intensity $h = \lambda = 3\%$) with gaussian copula

As expected, recovery rate acts as a scale-location parameter of the loss distribution, shifting the centre of the distribution to the left tail[35] and reducing the standard deviation of the loss distribution (in the case of increasing recovery rates). This clearly affects CDO prices increasing the required spread for junior tranches and lowering it for senior tranches.

[35] Under the assumption specified above, this claim can be easily proved by differentiating equation (48) with respect to the recovery rate R, leading to:

$$\frac{\partial \mathbb{E}\left[L(t)\right]}{\partial R} = -NA\left[1 - \left(1 + \frac{St}{1-R}\right)\exp\left(-\frac{St}{1-R}\right)\right],$$

which is non positive for $s > 0$ and $R < 1$.

5.5 Copula functions and loss distribution

So far the analysis developed in previous sections has been founded on the assumption that dependence among obligors can be fully described through a gaussian copula. Similarly to the basket default swap case, this is not always a suitable choice, because of the asymptotic independence in tail of this multivariate distribution function. This aspect clearly influence results, especially when dealing with CDOs, given the large number of entities referenced by the contract. With this regard, Student's t copula offers a convenient way for taking into account the occurrence of joint extreme events among obligors; nevertheless, this influence, as it will be shown, varies according tranches seniority. Figure 15 plots the CDO tranches estimated spreads for different values of correlation using the gaussian and the t copula with the same assumptions and data of § 5.4:

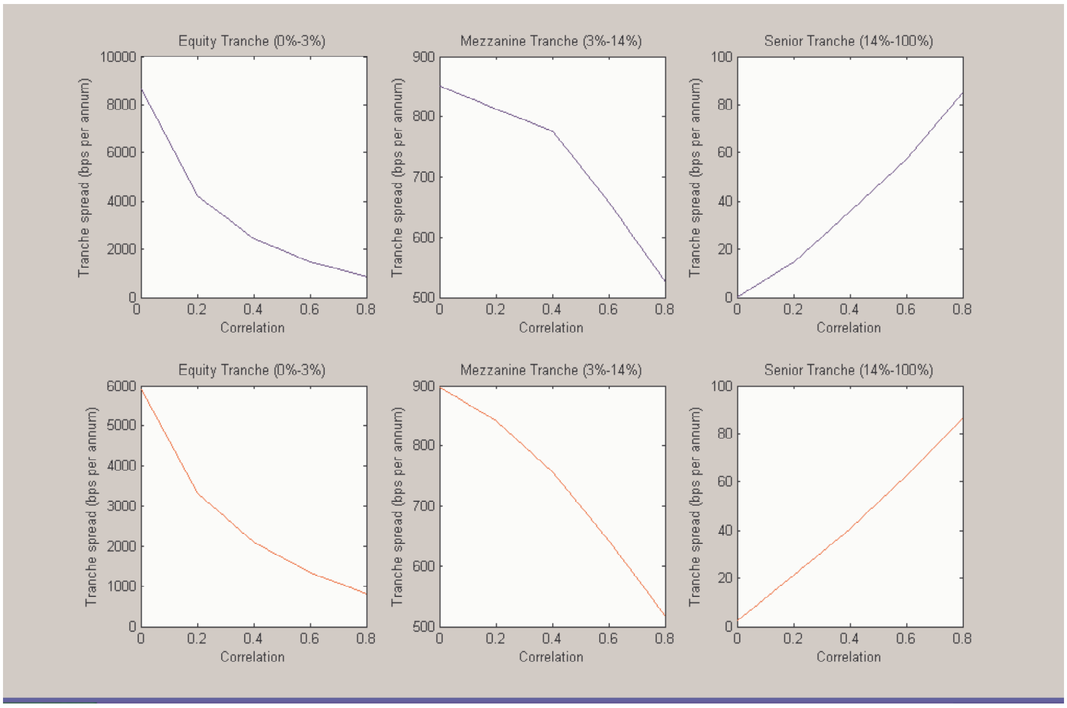

Figure 15. CDO tranches spreads priced with gaussian (above) and t_{10} (below) copula

Mezzanine tranche prices appear to be higher when computed through the t copula, with the opposite verified for the equity piece; senior tranches seem not to be particularly affected by the copula function chosen. This behavior can be reasonably explained by observing the loss distribution of the collateral pool. To this end, Figure 16 reports the defaults distribution estimated with the gaussian and the t copula for a collateral pool of $N = 100$ obligors with constant spread $S = 150bps$, recovery rate $R = 40\%$ with a time horizon $T = 5$:

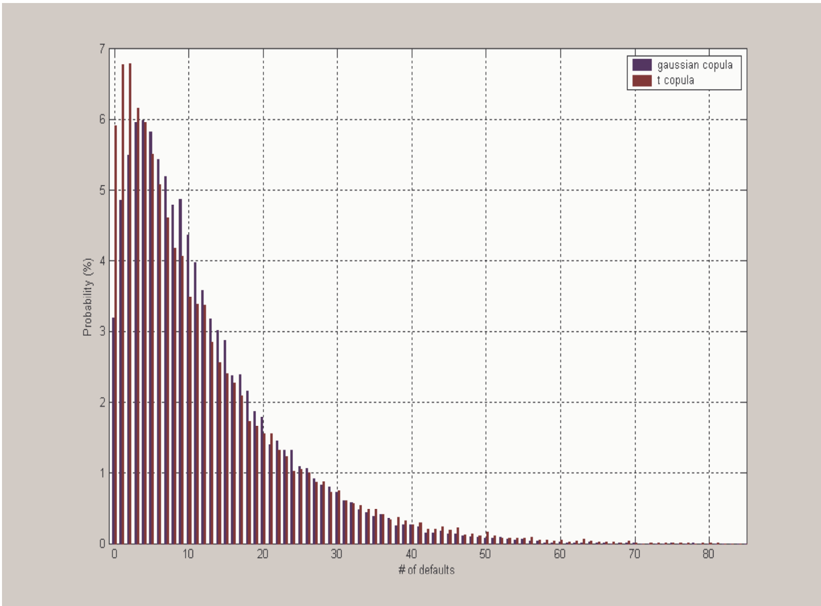

Figure 16. Collateral pool defaults distribution for the gaussian and t_{10} copula ($r = 0.2$)

As extensively pointed out before, the Student's t copula is characterized by fatter tails than the gaussian copula, inducing an higher probability on the left tail (less defaulted securities) and on the right tail (more defaulted obligors). Subsequently equity piece spreads will be lower with the t copula, taking advantage of the greater probability of less default events, while the opposite is verified for mezzanine tranches, which are penalized by the greater probability mass around the median of the distribution (more default events). On the contrary, senior tranches appear to be not particularly affected, showing the same price-correlation behavior for the two copula functions chosen to model dependency among obligors.

6 Conclusion

We have shown how copula functions can be embedded into a valuation model to asses prices of credit derivatives written on a portfolio of assets. The concept of asymptotic tail dependence as a measure of occurrence of joint extreme events has been introduced by stressing its impact in modeling the dependency structure in a reference pool of credits. To this end we have seen how t copula can efficiently captures the tail dependence among obligors by dealing better with the occurrence of joint default events. The problem of fitting copula parameters to observable market data has been addressed by presenting some of the methods currently employed in practice, with a specific application to a portfolio of four traded securities. We have then presented a simulation algorithm for valuing credit sensitive basket structures, whose consistency has been ensured by calibrating the parameters of the time to default univariate margins to market observable quotes. More precisely, by the conditioning property of Cox processes, we have provided a methodology to estimate the distribution of correlated default times and evaluate defaultable securities. We have then showed how Basket Default Swaps and Collateralized Debt Obligations can be valued within the above framework and we have analyzed how different factors (correlation, recovery rates, quality and granularity of the underlying portfolio) affect prices. As a general result, we have seen that t copula, compared with the gaussian, tends to underprice first to default contracts and CDO equity tranches, while the opposite is verified for second to defaults and mezzanine tranches; with regard to the former, we have showed that the difference between the prices computed with the two copulae, decreases with increasing correlation given the occurrence of more joint extreme events even in the gaussian framework. With regard to CDOs, we motivated this different behavior by looking at the loss distribution of the collateral pool, stressing the role played by the dependence in tail of the t copula in modeling expected losses.

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7 Appendix

7.1 Conditional probability density function for the Student's t distribution

We start from the expression for the bivariate (with linear correlation r and ν degrees of freedom) and univariate Student's t probability density function and apply the usual formula for computing the conditional density function:

$$
\begin{aligned}
f^{Student}(Y|X &= x) = \frac{f^{Student}(Y=y,X=x)}{f^{Student}(X=x)} \\
&= \frac{\dfrac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\nu\pi)\left(1-r^2\right)^{1/2}} \cdot \left(\frac{\nu\left(1-r^2\right)+x^2-2rxy+y^2}{\nu(1-r^2)}\right)^{-\left(\frac{\nu+2}{2}\right)}}{\dfrac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\nu\pi)^{1/2}} \cdot \dfrac{1}{(1+x^2/\nu)^{\frac{\nu+1}{2}}}} \\
&= \frac{\dfrac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)(\nu\pi)^{1/2}\left(1-r^2\right)^{1/2}} \cdot \left(\dfrac{\nu}{\nu+x^2}\right)^{1/2}}{\left[\dfrac{\nu\left(1-r^2\right)+x^2-2rxy+y^2+r^2x^2-r^2x^2}{\left(\nu+x^2\right)\left(1-r^2\right)}\right]^{\left(\frac{\nu+2}{2}\right)}} \\
&= \frac{\dfrac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)(\nu\pi)^{1/2}\left(1-r^2\right)^{1/2}} \cdot \left(\dfrac{\nu}{\nu+x^2}\right)^{1/2}}{\left[\dfrac{\nu\left(1-r^2\right)+x^2+(y-rx)^2-r^2x^2}{\left(\nu+x^2\right)\left(1-r^2\right)}\right]^{\left(\frac{\nu+2}{2}\right)}} \\
&= \frac{\dfrac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\left[(\nu+1)\pi\right]^{1/2}\left(1-r^2\right)^{1/2}}}{\left[1+\dfrac{(y-rx)^2}{\left(\nu+x^2\right)\left(1-r^2\right)}\right]^{\left(\frac{\nu+2}{2}\right)} \cdot \left(\dfrac{\nu+1}{\nu+x^2}\right)^{-1/2}} \\
&= \frac{\Gamma\left(\frac{(\nu+1)+1}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\left[(\nu+1)\pi\right]^{1/2}} \cdot \frac{\left[\dfrac{\nu+1}{\left(\nu+x^2\right)\left(1-r^2\right)}\right]^{1/2}}{\left[1+\frac{(y-rx)^2}{(\nu+x^2)(1-r^2)}\frac{\nu+1}{\nu+1}\right]^{\left(\frac{(\nu+1)+1}{2}\right)}}
\end{aligned}
$$

$$= \frac{\Gamma\left(\frac{(\nu+1)+1}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\left[(\nu+1)\pi\right]^{1/2}} \cdot \left[1 + \frac{\left(\frac{y-\mu}{\sigma}\right)^2}{\nu+1}\right]^{-\left(\frac{(\nu+1)+1}{2}\right)} \cdot \frac{1}{\sigma},$$

with $\mu = rx$ and $\sigma = \left[\frac{\left(\nu+x^2\right)\left(1-r^2\right)}{\nu+1}\right]^{1/2}$ and therefore follows that $Y|X = x$ is distributed as a Student's t random variable with parameters μ, σ and $\nu+1$.

7.2 Copula representation of the conditional probability of $U(0,1)$ variates

Consider a pair of uniform $(0,1)$ random variables (U,V) with copula C. We want to show that

$$\mathbb{P}(V \leq v | U = u) = \frac{\partial}{\partial u} C(u,v).$$

To this end, exploiting the right-continuity property of the distribution function, we have

$$\begin{aligned}
\mathbb{P}(V \leq v | U = u) &= \lim_{\Delta u \to 0} \mathbb{P}(V \leq v | u < U \leq u + \Delta u) \\
&= \lim_{\Delta u \to 0} \frac{\mathbb{P}(u < U \leq u + \Delta u, V \leq v)}{\mathbb{P}(u < U \leq u + \Delta u)}.
\end{aligned}$$

Since U is a $(0,1)$ random variable we have $\mathbb{P}(u < U \leq u + \Delta u) = \Delta u$, and we deduce that

$$\begin{aligned}
\mathbb{P}(V \leq v | U = u) &= \lim_{\Delta u \to 0} \frac{\mathbb{P}(u < U \leq u + \Delta u, V \leq v)}{\Delta u} \\
&= \lim_{\Delta u \to 0} \frac{C(u + \Delta u, v) - C(u,v)}{\Delta u} \\
&= \frac{\partial}{\partial u} C(u,v).
\end{aligned}$$