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11

QUESTIONS ON AMENABILITY

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I. SOME QUESTIONS ABOUT AMENABLE GROUPS

QUESTION 11.1. *Is it true that any infinite amenable group contains an infinite abelian subgroup ? (This is of course of interest only for torsion groups.)*

QUESTION 11.2. *For solvable non-virtually nilpotent groups, is there a canonical way of constructing a Følner sequence ? (Say, in terms of generators or judiciously chosen neighborhoods of the identity.)*

QUESTION 11.3. *Is there a nice characterization of amenability of a group G via the topological algebra in βG, the Stone–Čech compactification of G ? (Here the term "topological algebra" refers to properties of left or right ideals, idempotents, etc.)*

DEFINITION 11.4. A set $R \subseteq G \setminus \{e\}$ is said to have *property TR* (for Topological Recurrence) if for every minimal action of G by homeomorphisms $T_g, g \in G$ of a compact metric space X and any open non-empty set $U \subseteq X$ there exists $g \in R$ such that $U \cap T_g U \neq \varnothing$. Here *minimal* means that, for any $x \in X$, $\overline{\{T_g x, g \in G\}} = X$. A set $R \subseteq G \setminus \{e\}$ is said to have *property MR* (for Measurable Recurrence) if for any action of G by measure preserving transformations $T_g, g \in G$ on a probability space $(X, \mathcal{B}, \mu)$ and any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $g \in R$ such that $\mu(A \cap T_g A) > 0$.

CONJECTURE 11.5. *A countable discrete group is amenable if and only if property MR implies property TR (that is, every set of measurable recurrence is a set of topological recurrence).*

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There are amenable groups which are minimally almost periodic, i.e. have no non-trivial finite-dimensional unitary representations. In the language of ergodic theory this simply means that any *ergodic* finite measure preserving action of such a group is automatically *weakly mixing*. One more equivalent formulation of this property is the following (see [3] and [2], Theorems 3.2 and 1.9):

DEFINITION 11.6. An amenable group G is *minimally almost periodic* if for any unitary representation $(U_g)_{g\in G}$ on a Hilbert space $\mathcal{H}$, one has a G-invariant splitting $\mathcal{H} = \mathcal{H}_{inv} \oplus \mathcal{H}_{wm}$, where

$$\mathcal{H}_{inv} = \{ f \in \mathcal{H} : U_g f = f \ \forall g \in G\} \quad \text{and}$$

$$\mathcal{H}_{wm} = \{ f \in \mathcal{H} : \forall \epsilon > 0 \text{ the set } \{g \in G : |\langle U_g f, f\rangle| > \epsilon\} \text{ is neglectable} \},$$

where a set S is called *neglectable* if for any Følner sequence $(F_n)_{n\in\mathbf{N}}$ one has $\frac{|S\cap F_n|}{|F_n|} \to 0$ as $n \to \infty$.

QUESTION 11.7. *Are there countable discrete amenable groups for which the neglectable set appearing in the above splitting is always finite? In other words, are there amenable groups G possessing — similarly to, say,* SL$(2,\mathbf{R})$ *— the property of "decay of matrix coefficients" meaning that for any unitary action $U_g : \mathcal{H} \to \mathcal{H}$, $g \in G$ which has no invariant vectors, one has, for all $f \in \mathcal{H}$, $\langle U_g f, f\rangle \to 0$ as $g \to \infty$.*

II. SOME QUESTIONS ON INVARIANT MEANS

One of the many equivalent definitions of amenability for discrete groups is the postulation of the existence of invariant means on the Banach space $B_{\mathbf{R}}(G)$ of bounded real-valued functions on the group G. But even when G is non-amenable, certain important classes of functions on G possess an invariant and even unique mean. For example, by Ryll-Nardzewsky theorem [4], if G is any locally compact group, the space $WAP(G)$ of weakly almost periodic functions on G has a unique invariant mean. Since positive definite functions are weakly almost periodic, this implies that there exists a unique mean on the algebra of functions of the form $\varphi(g) = \langle U_g f_1, f_2\rangle$, where $U_g : \mathcal{H} \to \mathcal{H}, g \in G$ is a unitary representation of G on a Hilbert space $\mathcal{H}$ and $f_1, f_2 \in \mathcal{H}$. One can show (see for example [5]) that any such function φ can also be represented as $\varphi(g) = \int f_1(T_g x) f_2(x)\, d\mu(x)$ where $(T_g)_{g\in G}$ is a measure-preserving action of G on a probability space $(X, \mathcal{B}, \mu)$ and $f_1, f_2 \in L^\infty(X, \mathcal{B}, \mu)$. This makes natural the following question:

QUESTION 11.8. *Let G be a locally compact group, let $k \in \mathbf{N}$ and let $(T_g^{(1)})_{g\in G}, (T_g^{(2)})_{g\in G}, \ldots, (T_g^{(k)})_{g\in G}$ be k commuting measure-preserving actions of G on a probability space $(X, \mathcal{B}, \mu)$. ("Commuting" means that $T_g^{(i)}T_h^{(j)} = T_h^{(j)}T_g^{(i)}$ for $i \neq j$ and for all $g, h \in G$.) Is it true that there exists a unique invariant mean on the algebra of functions on G generated by the functions of the form*

$$\varphi(g) = \int f_0(x) f_1(T_g^{(1)}x) f_2(T_g^{(2)}T_g^{(1)}x) \ldots f_k(T_g^{(k)} \ldots T_g^{(2)}T_g^{(1)}x)\, d\mu\,,$$

where $f_i \in L^\infty(X, \mathcal{B}, \mu)$, $i = 0, 1, \ldots, k$.

REMARK 11.9. The answer is *yes* for $k = 1$ (as explained above) and, if G is amenable, for $k = 2$ (follows from [1]).

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