

■ Estimating Error in Riemann Sums

Recall that we can estimate the net signed area between a function $f(x)$ and the x -axis over some interval $[a, b]$ by a left or right Riemann sum:

$$\text{Net signed area} = \int_a^b f(x) dx \approx \begin{cases} \Delta x f(x_0) + \cdots + \Delta x f(x_{n-1}) & = L_n (\text{left Riemann sum}) \\ \text{or} \\ \Delta x f(x_1) + \cdots + \Delta x f(x_n) & = R_n (\text{right Riemann sum}) \end{cases}$$

For an increasing function the left and right sums are under and over estimates (respectively) and for a decreasing function the situation is reversed. In either case, we know that the actual net signed area must be between the two values. That is, for increasing functions we have:

$$\text{Left Riemann Sum} \leq \int_a^b f(x) dx \leq \text{Right Riemann Sum}$$

While for decreasing functions we instead have:

$$\text{Right Riemann Sum} \leq \int_a^b f(x) dx \leq \text{Left Riemann Sum}$$

You might want to make two sketches to convince yourself that this is the case.

The difference between the actual value of the definite integral and either the left or right Riemann sum is the **error** in that approximation. But in both of the cases above, this error can be no larger than the (absolute value of the) difference between the left and right sums, since one is an underestimate and the other is an overestimate. Thus we have:

$$\text{Error in either Riemann sum approximation} \leq |R_n - L_n|$$

To find a nice expression for the difference on the left, we just use the definitions and notice that most of the terms cancel:

$$\begin{aligned} [R_n] - [L_n] &= \left[\cancel{\Delta x f(x_1)} + \cancel{\Delta x f(x_2)} + \cdots + \cancel{\Delta x f(x_{n-1})} + \Delta x f(x_n) \right] \\ &\quad - \left[\Delta x f(x_0) + \cancel{\Delta x f(x_1)} + \cdots + \cancel{\Delta x f(x_{n-2})} + \cancel{\Delta x f(x_{n-1})} \right] \\ &= \Delta x (f(x_n) - f(x_0)) \\ &= \Delta x (f(b) - f(a)) \end{aligned}$$

Thus we have shown:

Theorem

$$\text{Right Sum} - \text{Left Sum} = \Delta x(f(b) - f(a))$$

But furthermore, since this is an upper bound for the error, we have the following result as well:

Theorem If $f(x)$ is increasing or decreasing on the interval $[a, b]$ then

$$\text{Error in Left or Right Riemann Sum} \leq \Delta x|f(b) - f(a)|.$$

► **Example**

How many subintervals in a Riemann sum will guarantee the approximate area under $f(x) = x^2 + 1$ on the interval $[0, 2]$ is accurate to 1 decimal place?

Solution We'll use a left Riemann sum to approximate. To make sure our approximation is correct to within 1 decimal place, we need the error in our approximation to be less than 0.05. Since $f(x)$ is increasing, the error will definitely be less than $\Delta x(f(b) - f(a))$, so we just need to make sure that

$$\Delta x(f(b) - f(a)) \leq 0.05$$

Since $\Delta x = (b - a)/n$ we need

$$\frac{b - a}{n}(f(b) - f(a)) \leq 0.05$$

↕

$$\frac{2 - 0}{n}(5 - 1) \leq 0.05$$

↕

$$\frac{8}{n} \leq 0.05$$

↕

$$n \geq 8/0.05 = 160 \text{ subintervals!}$$

Of course, our approximation might be within one decimal place with far fewer subintervals, but the point is that by using 160 we are *guaranteed* that the first decimal place will be correct.