1. Evaluate the integrals.

(a) $\int \frac{3x+2}{x^2+4}\,dx = \int \frac{3x}{x^2+4}\,dx + \int \frac{2}{x^2+4}\,dx = \frac{3}{2}\int \frac{2x\,dx}{x^2+4} + \int \frac{2\,dx}{x^2+4}$

$= \frac{3}{2}\int \frac{du}{u} + 2\cdot\left(\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right) + c$

$\left(\begin{array}{l} u = x^2+4 \\ du = 2x\,dx \end{array}\right)$

$= \boxed{\frac{3}{2}\ln|x^2+4| + \tan^{-1}\left(\frac{x}{2}\right) + c}$

(b) $\int_2^4 \frac{x^2+2}{x-1}\,dx$

Long division:

$$\begin{array}{r|l} & x+1 \\ \hline x-1 & x^2 + 0\cdot x + 2 \\ & -x^2 \oplus x \\ \hline & x+2 \\ & -x \oplus 1 \\ \hline & 3 \end{array}$$

$\therefore \frac{x^2+2}{x-1} = x+1+\frac{3}{(x-1)}$

$= \int_2^4 (x+1)\,dx + \int_2^4 \frac{3}{(x-1)}\,dx$

$= \frac{x^2}{2} + x + 3\ln|x-1|\Big]_2^4 = \frac{16}{2} + 4 + 3\ln(3) - \frac{4}{2} - 2 - 3\ln(2)$

$= \boxed{8 + 3\ln\left(\frac{3}{2}\right)}$

(c) $\int \frac{1-x}{1-\sqrt{x}}\,dx = \int \frac{\cancel{(1-\sqrt{x})}(1+\sqrt{x})}{\cancel{(1-\sqrt{x})}}\,dx = \int (1+x^{1/2})\,dx$

$= \cancel{x + \ldots}$ $= x + \frac{x^{3/2}}{\frac{3}{2}} + c = \boxed{x + \frac{2}{3}x^{3/2} + c}$

2. Evaluate the integrals.

(a) $\int 3x\sec^2(x)\,dx$

$u = 3x \qquad dv = \sec^2(x)\,dx$

$du = 3dx \qquad v = \tan(x)$

$= 3x\tan(x) - \int 3\tan(x)\,dx$

$= \boxed{3x\tan(x) - 3\ln|\sec(x)| + c}$

(b) $\int x^2 e^{4x}\,dx$

$u = x^2 \qquad dv = e^{4x}\,dx$

$du = 2x\,dx \qquad v = \frac{e^{4x}}{4}$

$= \frac{x^2 e^{4x}}{4} - \int 2x\frac{e^{4x}}{4}\,dx = \frac{x^2 e^{4x}}{4} - \frac{1}{2}\underbrace{\int xe^{4x}\,dx}$

$u = x \qquad dv = e^{4x}dx$

$du = dx \qquad v = \frac{e^{4x}}{4}$

$= \frac{x^2 e^{4x}}{4} - \frac{1}{2}\left[\frac{xe^{4x}}{4} - \int \frac{e^{4x}}{4}\,dx\right]$

$= \frac{x^2 e^{4x}}{4} - \frac{xe^{4x}}{8} + \frac{1}{8}\cdot\frac{e^{4x}}{4} + c$

$= \boxed{\frac{x^2 e^{4x}}{4} - \frac{xe^{4x}}{8} + \frac{e^{4x}}{32} + c}$

3. Evaluate the integrals.

(a) $\int x^7 \ln(5x)\,dx$

$u = \ln(5x)$ $\qquad$ $dv = x^7\,dx$

$du = \frac{1}{\cancel{5}x} \cdot \cancel{5}\,dx$ $\qquad$ $v = \frac{x^8}{8}$

$\parallel$

$$\frac{x^8}{8}\ln(5x) - \int \frac{1}{\cancel{x}} \cdot \frac{x^{\cancel{8}\,7}}{8}\,dx = \boxed{\frac{x^8}{8}\ln(5x) - \frac{x^8}{64} + C}$$

(b) $\int e^x \cos(3x)\,dx$

$u = \cos(3x)$ $\qquad$ $dv = e^x\,dx$

$du = -3\sin(3x)\,dx$ $\qquad$ $v = e^x$

$$\int e^x\cos(3x)\,dx = e^x\cos(3x) + \int 3\sin(3x)\,e^x\,dx$$

$$= e^x\cos(3x) + 3\underbrace{\int e^x\sin(3x)\,dx}$$

$u = \sin(3x)$

$du = 3\cos(3x)\,dx$

$dv = e^x\,dx$

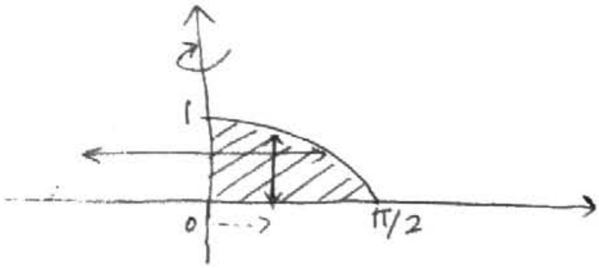
$v = e^x$

$$= e^x\cos(3x) + 3\left[e^x\sin(3x) - 3\int e^x\cos(3x)\,dx\right]$$

$$10\int e^x\cos(3x)\,dx = e^x\cos(3x) + 3e^x\sin(3x) + k$$

$$\therefore \int e^x\cos(3x)\,dx = \boxed{\frac{1}{10}\left(e^x\cos(3x) + 3e^x\sin(3x)\right) + c}$$

4. Find the volume of the solid generated when the region bounded by $y = \cos x$ and the x-axis on the interval $[0, \frac{\pi}{2}]$ is revolved about the y-axis.



Disk method:

$$\int_0^1 \pi(\cos^{-1}(y))\,dy$$

integration $\int \cos^{-1}(y)\,dy$ is similar to problem 5 below where we compute $\int \sin^{-1}(x)\,dx$

looks better than the above so we go by this method.

Shell method: $$2\pi \int_0^{\pi/2} x\cos(x)\,dx$$

we do integration by parts with

$$\begin{cases} u = x & dv = \cos(x)\,dx \\ du = dx & v = \sin(x) \end{cases}$$

$$= 2\pi\left[\, x\sin(x)\Big|_0^{\pi/2} - \int_0^{\pi/2} \sin(x)\,dx \right]$$

$$= 2\pi\left[\frac{\pi}{2}(1) + \cos(x)\Big|_0^{\pi/2}\right] = 2\pi\left[\frac{\pi}{2} + (0-1)\right]$$

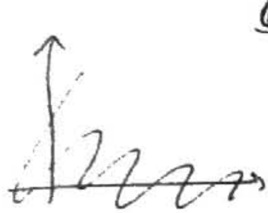
$$= \boxed{2\pi\left(\frac{\pi}{2} - 1\right)}$$

---

5. Find the area of the region bounded by the curves $y = \sin(x)$ and $y = \sin^{-1}(x)$ on the interval $[0, \frac{1}{2}]$.

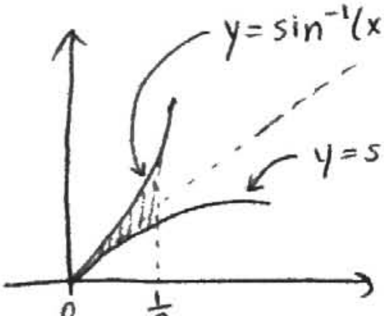
~~Points of intersection~~

~~$\sin(x) = \sin^{-1}(x)$~~



~~$\sin(\sin(x)) = x$~~

You need to remember that $y = \sin(x)$ lies below $y = x$ line. And $\sin^{-1}(x)$ is a reflection of $\sin(x)$ about $y = x$.



$$\int_0^{\frac{1}{2}} (\sin^{-1}(x) - \sin(x))\,dx$$

$$= \underbrace{\int_0^{\frac{1}{2}} \sin^{-1}(x)\,dx}_{\text{by parts}} - \int_0^{\frac{1}{2}} \sin(x)\,dx$$

by parts $u = \sin^{-1}(x)$ $dv = dx$

$$x\sin^{-1}(x)\Big|_0^{1/2} - \underbrace{\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx}_{\text{u sub } u = 1-x^2} + \cos(x)\Big|_0^{1/2}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) + \frac{1}{2}\int_1^{3/4} \frac{du}{u^{1/2}} + \cos\left(\frac{1}{2}\right) - 1$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) + \cancel{\frac{1}{2}}\,\frac{u^{1/2}}{\cancel{(1/2)}}\Big|_1^{3/4} + \cos\left(\frac{1}{2}\right) - 1$$

$$= \boxed{\frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) + \left(\frac{3}{4}\right)^{1/2} \cancel{-1} + \cos\left(\frac{1}{2}\right) \cancel{+1}}$$

6. Evaluate the integrals.

(a) $\int \sin^5(x)\,dx = \cancel{\int \sin^2(x)\cos^3(x)} \int \sin^4(x)\sin(x)\,dx$

$= \int (1-\cos^2 x)^2 \sin(x)\,dx$ Let $u = \cos(x)$

$\therefore du = -\sin(x)\,dx$

$= -\int (1-u^2)^2\,du = -\int (1 - 2u^2 + u^4)\,du$

$= -\frac{u^5}{5} + \frac{2u^3}{3} - u + C$

$= \boxed{-\frac{\cos^5(x)}{5} + \frac{2\cos^3(x)}{3} - \cos(x) + C}$

(b) $\int \cos^3(10x)\,dx = \int \cos^2(10x)\cos(10x)\,dx = \int (1-\sin^2(10x))\cos(10x)\,dx$

Let $u = \sin(10x)$ $\therefore du = 10\cos(10x)\,dx$

$\therefore \int \cos^3(10x)\,dx = \frac{1}{10}\int (1-u^2)\,du = \frac{1}{10}\left(u - \frac{u^3}{3}\right) + C$

$= \boxed{\frac{1}{10}\left(\sin(10x) - \frac{\sin^3(10x)}{3}\right) + C}$

(c) $\int \cos^2(10x)\,dx = \cancel{\frac{1}{2}\int 2\cos^2(10x)\,dx}$

$= \int \frac{1+\cos(20x)}{2}\,dx = \boxed{\frac{1}{2}x + \frac{\sin(20x)}{40} + C}$

Half-angle formula!

$2\cos^2\theta - 1 = \cos(2\theta)$

$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$

7. Evaluate the integrals.

(a) $\int 3\sin^5(x)\cos^8(x)\,dx$ $= \int 3\sin^4(x)\cos^8(x)\sin(x)\,dx$

Let $u = \cos(x)$

$\therefore du = -\sin(x)\,dx$

$$= 3\int (1-\cos^2(x))^2 \cos^8(x)\sin(x)\,dx$$

$$= -3\int (1-u^2)^2 u^8\,du = -3\int (1-2u^2+u^4)u^8\,du$$

$$= -3\int (u^8 - 16u^{10} + u^{12})\,du$$

$$= -3\left(\frac{u^9}{9} - \frac{16u^{11}}{11} + \frac{u^{13}}{13}\right) + c$$

$$= \boxed{-\frac{\cos^9(x)}{3} - \frac{16\cos^{11}(x)}{11} + \frac{\cos^{13}(x)}{13} + c}$$

(b) $\int \sin^{\pi}(x)\cos^3(x)\,dx$ $= \int \sin^{\pi}(x)\cos^2(x)\cos(x)\,dx$

Let $u = \sin(x)$

$\therefore du = \cos(x)\,dx$

$$= \int u^{\pi}\cdot(1-u^2)\cdot du$$

$$= \int (u^{\pi} - u^{\pi+2})\,du$$

$$= \frac{u^{\pi+1}}{\pi+1} - \frac{u^{\pi+3}}{\pi+3} + c$$

$$= \boxed{\frac{\sin^{\pi+1}(x)}{\pi+1} - \frac{\sin^{\pi+3}(x)}{\pi+3} + c}$$

8. Evaluate the integrals.

(a) $\int 7\sec^6(x)\tan^{10}(x)\,dx = 7\int \sec^4(x)\tan^{10}(x)\sec^2(x)\,dx$

$= 7\int (1+\tan^2(x))^2 \tan^{10}(x)\sec^2(x)\,dx$

let $u = \tan(x)$
$du = \sec^2(x)\,dx$

$= 7\int (1+u^2)^2 u^{10}\,du$

$= 7\int (1+2u^2+u^4)u^{10}\,du$

$= 7\int (u^{10} + 2u^{12} + u^{14})\,du$

$= 7\left(\frac{u^{11}}{11} + \frac{2u^{13}}{13} + \frac{u^{15}}{15}\right) + C$

$= \boxed{7\left(\frac{\tan^{11}(x)}{11} + \frac{2\tan^{13}(x)}{13} + \frac{\tan^{15}(x)}{15}\right) + C}$

(b) $\int \tan^3(x)\sec^e(x)\,dx = \int \tan^2(x)\sec^{e-1}(x)\tan(x)\sec(x)\,dx$

$= \int (\sec^2(x)-1)\sec^{e-1}(x)\underbrace{\sec(x)\tan(x)dx}$ let $u = \sec(x)$

$\therefore du = \sec(x)\tan(x)dx$

$= \int (u^2-1)u^{e-1}\cdot du$

$= \int (u^{e+1} - u^{e-1})\,du$

$= \frac{u^{e+2}}{e+2} - \frac{u^e}{e} + C$

$= \boxed{\frac{\sec^{e+2}(x)}{e+2} - \frac{\sec^e(x)}{e} + C}$

9. Evaluate the integrals.

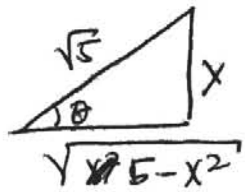
(a) $\int \frac{\sqrt{5-x^2}}{x}\,dx$ let $x = \sqrt{5}\sin\theta$ $\therefore dx = \sqrt{5}\cos\theta\, d\theta$ $\qquad \frac{x}{\sqrt{5}} = \sin\theta$

$$= \int \frac{\sqrt{5-5\sin^2\theta}}{\cancel{\sqrt{5}}\sin\theta} \cdot \cancel{\sqrt{5}}\cos(\theta)\, d\theta$$

$$= \sqrt{5}\int \frac{\cos(\theta)\cdot\cos(\theta)\, d\theta}{\sin(\theta)} = \sqrt{5}\int \frac{1-\sin^2(\theta)}{\sin(\theta)}\, d\theta$$

$$= \sqrt{5}\left(\int \csc\theta\, d\theta - \int \sin\theta\, d\theta\right)$$

$$= \sqrt{5}\left(-\ln\left|\csc\theta + \cot(\theta)\right| + \cos(\theta)\right) + C$$

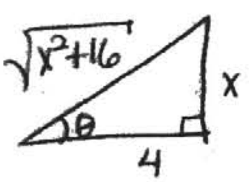


$$= \boxed{\sqrt{5}\left(-\ln\left|\frac{\sqrt{5}}{x} + \frac{\sqrt{5-x^2}}{x}\right| + \frac{\sqrt{5-x^2}}{\sqrt{5}}\right) + C}$$

(b) $\int \frac{1}{\sqrt{x^2+16}}\,dx$ Let $x = 4\tan\theta$ $\therefore dx = 4\sec^2\theta\, d\theta$ $\left(\frac{x}{4} = \tan\theta\right)$

$$= \int \frac{4\sec^2\theta\, d\theta}{\sqrt{16\tan^2\theta + 16}} = \int \frac{\cancel{4}\sec^{\cancel{2}}\theta\, d\theta}{\cancel{4}\cancel{\sec\theta}}$$

$$= \int \sec(\theta)\, d\theta = \ln\left|\sec(\theta) + \tan(\theta)\right| + C$$



$$= \boxed{\ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C}$$

10. Evaluate the integrals.

(a) $\displaystyle\int_3^{3\sqrt{2}} \frac{x}{\sqrt{x^2-9}}\,dx$

Let $x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

bounds:

$3 = 3\sec\theta \Rightarrow 1 = \sec\theta \Rightarrow \theta = 0$

$3\sqrt{2} = \cancel{3}\sec\theta \Rightarrow \sqrt{2} = \sec\theta$

$\Rightarrow \frac{1}{\sqrt{2}} = \cos\theta \Rightarrow \theta = \frac{\pi}{4}$

$$= \int_0^{\pi/4} \frac{3\sec\theta \cdot 3\sec\theta\tan\theta\,d\theta}{\sqrt{9\sec^2\theta - 9}}$$

$$= \int_0^{\pi/4} \frac{9\,\cancel{3}\sec^2\theta\,\cancel{\tan\theta}}{3\,\cancel{3}\,\cancel{\tan\theta}}\,d\theta = 3\tan\theta\Big]_0^{\pi/4} = 3\tan\left(\tfrac{\pi}{4}\right) - 3\tan(0)$$

$$= \cancel{1-0}\ \ 3-0$$

$$= \boxed{\cancel{1}}\ \ 3$$

(b) $\displaystyle\int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\,dx$

Let

$x = 2\sin\theta$

$\therefore dx = 2\cos\theta\,d\theta$

bounds:

$1 = 2\sin\theta$

$\therefore \frac{1}{2} = \sin\theta \Rightarrow \theta = \frac{\pi}{6}$

$\sqrt{2} = 2\sin\theta$

$\therefore \frac{1}{\sqrt{2}} = \sin\theta \Rightarrow \theta = \frac{\pi}{4}$

$$\int_{\pi/6}^{\pi/4} \frac{2\cos\theta\,d\theta}{4\sin^2\theta\sqrt{4-4\sin^2\theta}}$$

$$= \int_{\pi/6}^{\pi/4} \frac{\cancel{2}\cdot\cancel{\cos\theta}\,d\theta}{2\sin^2\theta\cdot 2\cdot\cancel{\cos\theta}} = \frac{1}{4}\int_{\pi/6}^{\pi/4} \csc^2(\theta)\,d\theta$$

$$= -\frac{1}{4}\left(\cot\theta\right)\Big|_{\pi/6}^{\pi/4} = -\frac{1}{4}\cdot\left(1-\sqrt{3}\right)$$

$$= \boxed{\frac{\sqrt{3}-1}{4}}$$

11. Write the appropriate form of the partial fraction decomposition for each of the following functions. Do not solve for the unknown constants.

(a) $\frac{5}{x^3 - 6x - 9x}$ ← this is a typo. It should be $\frac{5}{x^3 - 6x^2 + 9x}$

$$= \frac{5}{x\underbrace{(x^2 - 6x + 9)}_{\cancel{\text{irreducible}}}}$$

$$= \frac{5}{x(x-3)^2} \cancel{=} \cancel{\frac{A}{x} + \frac{Bx+C}{(x-3)} + \frac{Dx+E}{(x-3)}}$$

$$= \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

(b) $$\frac{x^2 - 1}{(x-1)^2(x+1)\underbrace{(x^2+1)}_{\text{irreducible}}} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)} + \frac{Dx+E}{x^2+1}$$

(c) $$\frac{5}{x^2(x^2 - x - 12)^3(x^2 + 3x + 4)^2} = \frac{5}{x^2(x-4)^3(x+3)^3\underbrace{(x^2+3x+4)^2}_{\text{irreducible}}}$$

$$= \cancel{\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x-4} + \frac{Ex+F}{}}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4} + \frac{D}{(x-4)^2} + \frac{E}{(x-4)^3} + \frac{F}{(x+3)} + \frac{G}{(x+3)^2} + \frac{H}{(x+3)^3}$$

$$+ \frac{Ix + J}{x^2 + 3x + 4} + \frac{Kx + L}{(x^2 + 3x + 4)^2}$$

(If the English alphabet is not enough use $A_1, A_2, A_3, \dots$)

12. Evaluate the integrals.

(a) $\int \frac{21x^2}{x^3 - x^2 - 12x}\,dx = \int \frac{21x^{\cancel{2}}}{\cancel{x}(x^2 - x - 12)}\,dx$

$$= \int \frac{21x}{(x-4)(x+3)}\,dx$$

Partial fraction decomp.:

$$\frac{21x}{(x-4)(x+3)} = \frac{A}{(x-4)} + \frac{B}{(x+3)} = \frac{A(x+3) + B(x-4)}{(x-4)(x+3)}$$

~~Set x = -B~~ $\quad \therefore\ 21x = A(x+3) + B(x-4)$

Set $x = -3$, $\cancel{-}21(3) = B(\cancel{-}7)$

$\therefore \boxed{B = 9}$

Set $x = 4$

$\cancel{3}21(4) = A(7) \Rightarrow \boxed{A = 12}$

$$\therefore \int \frac{21x}{(x-4)(x+3)}\,dx = \int \frac{12}{(x-4)}\,dx + \int \frac{9}{(x+3)}\,dx = \boxed{12\ln|x-4| + 9\ln|x+3| + C}$$

(b) $\int \frac{x^2 + x + 2}{(x+1)(x^2+1)}\,dx$

Partial fraction:

$$\frac{x^2+x+2}{(x+1)(x^2+1)} = \frac{A}{(x+1)} + \frac{Bx + C}{x^2+1}$$

$$\therefore\ x^2 + x + 2 = A(x^2+1) + (Bx+C)(x+1)$$

$$= Ax^2 + A + Bx^2 + Bx + Cx + C$$

$$= (A+B)x^2 + (B+C)x + (A+C)$$

$\therefore\ A + B = 1 \Rightarrow B = 1 - A$ — ①

$B + C = 1$ — ③

$A + C = 2 \Rightarrow C = 2 - A$ — ②

Plug ① and ② in ③

$1 - A + 2 - A = 1 \Rightarrow 3 - 2A = 1$

$\Rightarrow 2 = 2A$

$\Rightarrow A = 1$

From ①: $B = 0$

From ②: $C = 1$

$$= \int \left(\frac{1}{x+1} + \frac{1}{x^2+1}\right)dx$$

$$= \boxed{\ln|x+1| + \tan^{-1}(x) + C}$$

13. Evaluate each of the following integrals or state that the integral diverges.

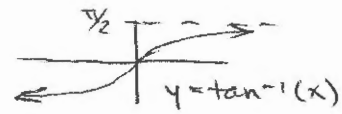
(a) $\displaystyle\int_1^{\infty} \frac{\tan^{-1}(x)}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{\tan^{-1}(x)}{1+x^2}\,dx$

$u = \tan^{-1}(x)$

$du = \dfrac{1}{1+x^2}\,dx$

@ $x=1$, $u = \tan^{-1}(1) = \dfrac{\pi}{4}$

@ $x=b$, $u = \tan^{-1}(b)$



$= \displaystyle\lim_{b\to\infty} \int_{\pi/4}^{\tan^{-1}(b)} u\,du$

$= \displaystyle\lim_{b\to\infty} \left[\tfrac{1}{2}u^2\right]_{\pi/4}^{\tan^{-1}(b)}$

$= \displaystyle\lim_{b\to\infty} \left[\tfrac{1}{2}(\tan^{-1}(b))^2 - \tfrac{1}{2}\left(\tfrac{\pi}{4}\right)^2\right]$

$= \tfrac{1}{2}\left(\tfrac{\pi}{2}\right)^2 - \tfrac{1}{2}\left(\tfrac{\pi}{4}\right)^2 = \dfrac{3\pi^2}{32}$

(b) $\displaystyle\int_{-\infty}^{1} e^{-4x}\,dx$

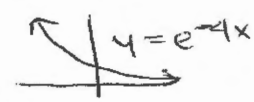
$= \displaystyle\lim_{a\to-\infty} \int_a^1 e^{-4x}\,dx = \lim_{a\to-\infty} \left[-\tfrac{1}{4}e^{-4x}\right]_a^1$

$= \displaystyle\lim_{a\to-\infty} \left[\left(-\tfrac{1}{4}e^{-4}\right) - \left(-\tfrac{1}{4}e^{-4a}\right)\right]$

$= \displaystyle\lim_{a\to-\infty} \left(\tfrac{1}{4}e^{-4a} - \tfrac{1}{4}e^{-4}\right)$

note: $\displaystyle\lim_{a\to-\infty} e^{-4a} = \infty$

$= \infty$ <u>diverges</u> (to ∞)



(c) $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_0^{\infty} xe^{-x^2}\,dx$

$= \displaystyle\lim_{a\to-\infty} \int_a^0 xe^{-x^2}\,dx + \lim_{b\to\infty} \int_0^b xe^{-x^2}\,dx$

$= \displaystyle\lim_{a\to-\infty} \left[-\tfrac{1}{2}e^{-x^2}\right]_a^0 + \lim_{b\to\infty} \left[-\tfrac{1}{2}e^{-x^2}\right]_0^b$

$= \displaystyle\lim_{a\to-\infty} \left[\left(-\tfrac{1}{2}\right) - \left(-\tfrac{1}{2}e^{-a^2}\right)\right] + \lim_{b\to\infty} \left[-\tfrac{1}{2}e^{-b^2} - \left(-\tfrac{1}{2}\right)\right]$

$= \left(-\tfrac{1}{2}\right) - 0 + 0 - \left(-\tfrac{1}{2}\right)$

$= 0$

14. Evaluate each of the following integrals or state that the integral diverges.

(a) $\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$ $= \lim_{a\to 0^+} \int_a^8 x^{-1/3}\,dx$

$= \lim_{a\to 0^+} \left[\frac{3}{2} x^{2/3} \right]_a^8$

$= \lim_{a\to 0^+} \left[6 - \frac{3}{2} a^{2/3} \right]$

$= 6 - 0$

$= 6$

(b) $\int_{-\frac{\pi}{2}}^0 \tan(x)\,dx$ $= \lim_{a\to -\frac{\pi}{2}^+} \int_a^0 \tan(x)\,dx$

$= \lim_{a\to -\frac{\pi}{2}^+} \left[-\ln|\cos x| \right]_a^0$

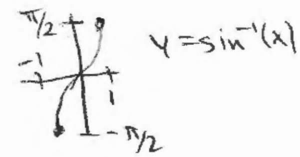
$= \lim_{a\to -\frac{\pi}{2}^+} \left[0 - (-\ln|\cos a|) \right]$

$= \lim_{a\to -\frac{\pi}{2}^+} \ln|\cos a|$

$= -\infty$ <u>diverges</u> (to $-\infty$)

(c) $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx$

$= \int_{-1}^0 \frac{1}{\sqrt{1-x^2}}\,dx + \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$



$= \lim_{a\to -1^+} \int_a^0 \frac{1}{\sqrt{1-x^2}}\,dx + \lim_{b\to 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}}\,dx$

$= \lim_{a\to -1^+} \left[\sin^{-1}(x) \right]_a^0 + \lim_{b\to 1^-} \left[\sin^{-1}(x) \right]_0^b$

$= \lim_{a\to -1^+} \left[0 - \sin^{-1}(a) \right] + \lim_{b\to 1^-} \left[\sin^{-1}(b) - 0 \right]$

$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$

15. Evaluate each of the following integrals or state that the integral diverges.

(a) $\int_0^9 \frac{1}{(x-1)^{\frac{1}{3}}}\,dx = \int_0^1 \frac{1}{(x-1)^{1/3}}\,dx + \int_1^9 \frac{1}{(x-1)^{1/3}}\,dx$

$= \lim_{b\to 1^-} \int_0^b (x-1)^{-1/3}\,dx + \lim_{a\to 1^+} \int_a^9 (x-1)^{-1/3}\,dx$

$= \lim_{b\to 1^-} \left[\frac{3}{2}(x-1)^{2/3}\right]_0^b + \lim_{a\to 1^+} \left[\frac{3}{2}(x-1)^{2/3}\right]_a^9$

$= \lim_{b\to 1^-} \left[\frac{3}{2}(b-1)^{2/3} - \frac{3}{2}\right] + \lim_{a\to 1^+} \left[6 - \frac{3}{2}(a-1)^{2/3}\right]$

$= 0 - \frac{3}{2} + 6 - 0$

$= \frac{9}{2}$

(b) $\int_{-1}^2 \frac{36}{x^3}\sqrt{\frac{12}{x^2}+13}\,dx = \int_{-1}^0 \frac{36}{x^3}\sqrt{\frac{12}{x^2}+13}\,dx + \int_0^2 \frac{36}{x^3}\sqrt{\frac{12}{x^2}+13}\,dx$

$= \lim_{b\to 0^-} \int_{-1}^b 36x^{-3}\sqrt{12x^{-2}+13}\,dx + \lim_{a\to 0^+} \int_a^2 36x^{-3}\sqrt{12x^{-2}+13}\,dx$

$= \lim_{b\to 0^-} \int_{25}^{\frac{12}{b^2}+13} -\frac{3}{2}\sqrt{u}\,du + \lim_{a\to 0^+} \int_{\frac{12}{a^2}+13}^{16} -\frac{3}{2}\sqrt{u}\,du$

$u = 12x^{-2}+13$

$du = -24x^{-3}\,dx$

$-\frac{1}{24}\,du = x^{-3}\,dx$

@ $x=-1$, $u=25$

@ $x=b$, $u=\frac{12}{b^2}+13$

@ $x=a$, $u=\frac{12}{a^2}+13$

@ $x=2$, $u=16$

$= \lim_{b\to 0^-} \left[-u^{3/2}\right]_{25}^{\frac{12}{b^2}+13} + \lim_{a\to 0^+} \left[-u^{3/2}\right]_{\frac{12}{a^2}+13}^{16}$

$= \lim_{b\to 0^-} \left[-\left(\frac{12}{b^2}+13\right)^{3/2} - \left(-25^{3/2}\right)\right]$

$+ \lim_{a\to 0^+} \left[-16^{3/2} - \left(-\left(\frac{12}{a^2}+13\right)^{3/2}\right)\right]$

diverges because $\lim_{b\to 0^-} \left[-\left(\frac{12}{b^2}+13\right)^{3/2} + 125\right] = -\infty$

(and/or because $\lim_{a\to 0^+} \left[-64 + \left(\frac{12}{a^2}+13\right)^{3/2}\right] = \infty$)