Math 1161: Written Homework 6 Name: ______________________ .# ________

Due 14 November 2017 in recitation. TA: ______________________ Time: ________

Instructions. You may discuss this assignment with others, but you must submit your own write-up. Write clearly and legibly. All functions herein are real-valued functions of a single real variable.

1. (12 pts) To find areas and volumes in calculus, our work-flow may be summarized as: *slice*, *approximate*, *integrate*.

   When performing the "approximate" step, it is common to hear mathematicians or physicists make comments like "only the dx terms matter" or "(dx)^2 is much smaller than dx, so we can ignore the (dx)^2 terms". In this context, dx means a small change in x. To help us make sense of these comments formally, we will use Δx to represent a small change in x, and we will consider the Shell Method as an instructive example.

   Suppose f is a positive continuous function on $[a, b]$, where $0 \le a$. (So, the graph of f is in the first quadrant.)

   Let $\mathcal{R}$ be the region between the graph of f and the x-axis on the interval $[a, b]$, and let $\mathcal{S}$ be the solid of revolution formed by revolving $\mathcal{R}$ about the y-axis.

   When using the Shell Method to find the volume V of $\mathcal{S}$, we can first "slice" $\mathcal{R}$ vertically and approximate it using n rectangles that correspond to a Left Riemann Sum of f on $[a, b]$. Then, we can revolve each of these rectangles about the y-axis to get shells. (These shells are the "slices" of our volume.)

   We can write an exact formula for the volume V_k of the k-th shell by thinking of it as "an outer cylinder minus an inner cylinder". Because the k-th shell is formed by revolving the k-th rectangle about the y-axis, its inner radius is x_{k-1}, its outer radius is x_k, and its height is $f(x_{k-1})$. So,

$$V_k = \pi x_k^2 f(x_{k-1}) - \pi x_{k-1}^2 f(x_{k-1})\,.$$

   Now, $x_k = x_{k-1} + \Delta x$, where $\Delta x = x_k - x_{k-1} = \frac{b-a}{n}$, because Left Riemann Sums use regular partitions. So, we can rewrite

$$\begin{aligned}
V_k &= \pi(x_{k-1} + \Delta x)^2 f(x_{k-1}) - \pi x_{k-1}^2 f(x_{k-1}) \\
&= \pi\big(x_{k-1}^2 + 2x_{k-1}\Delta x + (\Delta x)^2\big) f(x_{k-1}) - \pi x_{k-1}^2 f(x_{k-1}) \\
&= \pi x_{k-1}^2 f(x_{k-1}) + 2\pi x_{k-1} f(x_{k-1})\Delta x + \pi f(x_{k-1})(\Delta x)^2 - \pi x_{k-1}^2 f(x_{k-1}) \\
&= 2\pi x_{k-1} f(x_{k-1})\Delta x + \pi f(x_{k-1})(\Delta x)^2
\end{aligned}$$

   If we ignore the $(\Delta x)^2$ term, we obtain the approximation

$$V_k \approx 2\pi x_{k-1} f(x_{k-1})\Delta x\,.$$

   A mathematician or physicist may justify ignoring the $(\Delta x)^2$ term by saying something like "this approximation becomes precise in the limit," so

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} V_k = \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi x_{k-1} f(x_{k-1})\Delta x\,.$$

   Then, by recognizing $\sum_{k=1}^{n} 2\pi x_{k-1} f(x_{k-1})\Delta x$ as a Left Riemann Sum of the function $2\pi x f(x)$ on the interval $[a, b]$, it follows that

$$V = \int_a^b 2\pi x f(x)\,\mathrm{d}x\,,$$

   which is the formula from the textbook.

   This exercise will justify why we can ignore the $(\Delta x)^2$ term.

(continued on reverse)

(a) Show that, for any constant C and for regular partitions of $[a, b]$ into n subintervals, $\lim_{n\to\infty} \sum_{k=1}^{n} C(\Delta x)^2 = 0$.

(b) Use part (a) to show that $\lim_{n\to\infty} \sum_{k=1}^{n} \pi f(x_{k-1})(\Delta x)^2 = 0$.

Hint: By the Extreme Value Theorem, f has a minimum m and a maximum M on $[a, b]$. Now, recall the Squeeze Theorem.

(c) The function $2\pi x f(x)$ is continuous on $[a, b]$, so it is integrable on $[a, b]$. So, the limit $\lim_{n\to\infty} \sum_{k=1}^{n} 2\pi x_{k-1} f(x_{k-1})\Delta x$ exists because it is a Left Riemann Sum of $2\pi x f(x)$ on $[a, b]$.

Use limit laws and summation laws to show that $\lim_{n\to\infty} \sum_{k=1}^{n} V_k$ exists and is equal to $\lim_{n\to\infty} \sum_{k=1}^{n} 2\pi x_{k-1} f(x_{k-1})\Delta x$.

(Use the exact formula for V_k here.)

2. (8 pts) Two frustrums of cones are fused together on their narrow ends as pictured. Find the volume of the resulting shape.

   The wide radius of the top frustrum is 3 m. The wide radius of the bottom frustrum is 4 m. The narrow radius of both frustrums is 2 m. The height of the top frustrum is 6 m. The height of the bottom frustrum is 10 m.