Lecture 2 – August 24, 2018

- Online Homework 1.3, 1.4 and 2.2 now due Tuesday 8/28
- Written Homework 1 posted on
  `https://people.math.osu.edu/broaddus.9/11610X/`

Today

- Classes of Functions
- Limit Laws
- Applying Limit Laws
- What to do when plugging in fails
- The Squeeze Theorem

Classes of Functions

Constant functions function of the form $f(x) = c$ where c is a constant. e.g. $f(x) = 3$, $g(x) = \sin(\pi^2 + 8)$

Linear functions functions of the form $f(x) = ax + b$ where a and b are constants. e.g. $f(y) = -6y + \frac{1}{2}$, $h(r) = r\cos(\pi + 1)$

Quadratics *(why not bidratics?)* functions of the form $f(x) = ax^2 + bx + c$ where a, b, c are constants with $a \neq 0$.

Polynomials functions of the form $p(x) = a_0 + a_1x + \cdots + a_nx^n$. If $a_n \neq 0$ then we say the polynomial p has **degree** n.

Rational functions functions of the form $f(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials.

Limit Laws

- Recall from Wednesday

$$\sin(u + v) \neq \sin(u) + \sin(v)$$

$$\sin(uv) \neq \sin(u)\sin(v)$$

- **Limit Laws** say you **can** do this (and more) with limits.

Limit Laws

Theorem (Atomic Limits)

- $\lim_{x \to a} c = c$
- $\lim_{x \to a} x = a$

Theorem (Limit Laws)

If $\lim_{x \to a} f(x)$ *and* $\lim_{x \to a} g(x)$ *exist then*

Const. Mult. $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$

Sum Rule $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

Product Rule $\lim_{x \to a}(f(x) \cdot g(x)) = (\lim_{x \to a} f(x)) (\lim_{x \to a} g(x))$

Quotient Rule $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ **if** $\lim_{x \to a} g(x) \neq 0$

Power Rule $\lim_{x \to a}(f(x))^n = (\lim_{x \to a} f(x))^n$

Fract. Power Rule $\lim_{x \to a}(f(x))^{\frac{m}{n}} = (\lim_{x \to a} f(x))^{\frac{m}{n}}$ *if* $f(x) \geq 0$ *for* x *near* a.

Mystery Rule *What else would be helpful here????*

Applying Limit Laws

Example (Applying Limit Laws)

- Compute $\lim_{x \to 3}(x^3 - 4x^2 + 6)$ using only Atomic Limits and Limit Laws.
- $$\begin{aligned}
\lim_{x \to 3}(x^3 - 4x^2 + 6) &= \lim_{x \to 3} x^3 - \lim_{x \to 3} 4x^2 + \lim_{x \to 3} 6 \\
&= \lim_{x \to 3} x^3 - \lim_{x \to 3} 4x^2 + \lim_{x \to 3} 6 \\
&= \left(\lim_{x \to 3} x\right)^3 - 4\left(\lim_{x \to 3} x^2\right) + 6 \\
&= (3)^3 - 4\left(\lim_{x \to 3} x\right)^2 + 6 \\
&= (3)^3 - 4(3)^2 + 6 \\
&= -3
\end{aligned}$$

Applying Limit Laws

- In example above looks like we could have just plugged in 3 for x
- When is this ok?
- Exact answer soon
- For now, following theorem says that plugging in is ok if f is a rational function.

Theorem (Limits of polynomials and rational functions)

If p and q are polynomials then

1. $\lim_{x \to a} p(x) = p(a)$
2. $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ *assuming* $q(a) \neq 0$

What to do when plugging in fails

- What if plugging in fails?

Helpful techniques when plugging in fails

1. Factor and cancel
2. Put fractions over common denominators
3. Use algebraic conjugates

Theorem (Functions differing at single point have same limits)

If $f(x) = g(x)$ for all x near a (except possibly at $x = a$) and $\lim_{x \to a} f(x)$ *exists then* $\lim_{x \to a} g(x)$ *exists and*

$$\lim_{x \to a} g(x) = \lim_{x \to a} f(x).$$

What to do when plugging in fails

Example (Factor and cancel)

- Compute $\lim_{x\to 2} \frac{x^2-4}{x-2}$

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2-4}{x-2} &= \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} \\
&= \lim_{x\to 2} \begin{cases} x+2, & x \neq 2 \\ \text{undefined}, & x = 2 \end{cases} \\
&= \lim_{x\to 2} x+2 \\
&= 4
\end{aligned}$$

Example (Put fractions over common denominators)

- Compute $\lim_{x\to 3} \frac{\frac{1}{x+2}-\frac{1}{5}}{x-3}$

$$\begin{aligned}
\lim_{x\to 3} \frac{\frac{1}{x+2}-\frac{1}{5}}{x-3} &= \lim_{x\to 3} \frac{\frac{5}{5x+10}-\frac{x+2}{5x+10}}{x-3} \\
&= \lim_{x\to 3} \frac{5-x-2}{(5x+10)(x-3)} \\
&= \lim_{x\to 3} \frac{-x+3}{(5x+10)(x-3)} \\
&= \lim_{x\to 3} \frac{-1}{(5x+10)} \\
&= -\frac{1}{25}
\end{aligned}$$

What to do when plugging in fails

Example (Multiply by algebraic conjugate)

- Compute $\lim_{x\to 3} \frac{x-3}{\sqrt{x+1}-2}$

$$
\begin{aligned}
\lim_{x\to 3} \frac{x-3}{\sqrt{x+1}-2} &= \lim_{x\to 1} \frac{x-3}{\sqrt{x+1}-2} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} \\
&= \lim_{x\to 3} \frac{(x-3)(\sqrt{x+1}+2)}{x-3} \\
&= \lim_{x\to 3} \sqrt{x+1}+2 \\
&= \lim_{x\to 3} \sqrt{x+1} + \lim_{x\to 3} 2 \\
&= \sqrt{\lim_{x\to 3} x+1} + 2 \\
&= \sqrt{4}+2 \\
&= 4
\end{aligned}
$$

Squeeze Theorem

Theorem (Squeeze Theorem)

If

$$f(x) \leq g(x) \leq h(x)$$

for all x near a (except possibly at $x = a$) and

$$\lim_{x\to a} f = \lim_{x\to a} h = L$$

then

$$\lim_{x\to a} g(x) = L$$

Squeeze Theorem

Example (Using Squeeze Theorem)

- Let
$$g(x) = 3 + (x - 10)\sin(x)$$
Compute $\lim_{x \to 10} g(x)$.
- $3 - |x - 10| \leq g(x) \leq 3 + |x - 10|$ and $\lim_{x \to 10} 3 - |x - 10| = \lim_{x \to 10} 3 + |x - 10| = 3$
- Thus $\lim_{x \to 10} g(x) = 3$.