Lecture 3 – August 27, 2018

- Online Homework 1.3, 1.4, 2.2, 2.3, and 2.4 due Tuesday 8/28
- Written Homework 1 posted on
  `https://people.math.osu.edu/broaddus.9/11610X/`

Today

- What to do when plugging in fails
- Infinite limits
- Vertical asymptotes

What to do when plugging in fails

Example (Factor and cancel)

- Compute $\lim_{x\to 2} \frac{x^2-4}{x-2}$

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2-4}{x-2} &= \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} \\
&= \lim_{x\to 2} \begin{cases} x+2, & x \neq 2 \\ \text{undefined}, & x = 2 \end{cases} \\
&= \lim_{x\to 2} x+2 \\
&= 4
\end{aligned}$$

Example (Put fractions over common denominators)

- Compute $\lim_{x\to 3} \frac{\frac{1}{x+2}-\frac{1}{5}}{x-3}$

$$\begin{aligned}
\lim_{x\to 3} \frac{\frac{1}{x+2}-\frac{1}{5}}{x-3} &= \lim_{x\to 3} \frac{\frac{5}{5x+10}-\frac{x+2}{5x+10}}{x-3} \\
&= \lim_{x\to 3} \frac{5-x-2}{(5x+10)(x-3)} \\
&= \lim_{x\to 3} \frac{-x+3}{(5x+10)(x-3)} \\
&= \lim_{x\to 3} \frac{-1}{(5x+10)} \\
&= -\frac{1}{25}
\end{aligned}$$

What to do when plugging in fails

Example (Multiply by algebraic conjugate)

- Compute $\lim_{x\to 3} \frac{x-3}{\sqrt{x+1}-2}$

$$\begin{aligned}
\lim_{x\to 3} \frac{x-3}{\sqrt{x+1}-2} &= \lim_{x\to 1} \frac{x-3}{\sqrt{x+1}-2} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} \\
&= \lim_{x\to 3} \frac{(x-3)(\sqrt{x+1}+2)}{x-3} \\
&= \lim_{x\to 3} \sqrt{x+1}+2 \\
&= \lim_{x\to 3} \sqrt{x+1} + \lim_{x\to 3} 2 \\
&= \sqrt{\lim_{x\to 3} x+1} + 2 \\
&= \sqrt{4}+2 \\
&= 4
\end{aligned}$$

Squeeze Theorem

Example (Using Squeeze Theorem)

Let

$$g(x) = 3 + (x-10)\sin(x)$$

Compute $\lim_{x\to 10} g(x)$.
Solution:

- **Claim A:** For all x

$$3 - |x-10| \leq 3 + (x-10)\sin x \leq 3 + |x-10|$$

- Suppose $(x-10) \geq 0$. Then

$$\begin{gathered}
-1 \leq \sin x \leq 1 \\
-|x-10| \leq |x-10|\sin x \leq |x-10| \\
3 - |x-10| \leq 3 + |x-10|\sin x \leq 3 + |x-10| \\
3 - |x-10| \leq 3 + (x-10)\sin x \leq 3 + |x-10|
\end{gathered}$$

Example (Using Squeeze Theorem *(continued)*)

- Suppose $(x-10)<0$. Then

$$\begin{gathered}-1 \leq -\sin x \leq 1 \\ -|x-10| \leq -|x-10|\sin x \leq |x-10| \\ 3-|x-10| \leq 3-|x-10|\sin x \leq 3+|x-10| \\ 3-|x-10| \leq 3-(-x+10)\sin x \leq 3+|x-10| \\ 3-|x-10| \leq 3+(x-10)\sin x \leq 3+|x-10|\end{gathered}$$

Thus for all x we may conclude

$$3-|x-10| \leq 3+(x-10)\sin x \leq 3+|x-10|$$

Example (Using Squeeze Theorem *(continued)*)

- **Claim B:** $\lim_{x\to 10} 3-|x-10| = 3$
-

$$\lim_{x\to 10^+} 3-|x-10| = \lim_{x\to 10^+} 3-(x-10) = \lim_{x\to 10^+} -x+13 = -10+13 = 3$$
$$\lim_{x\to 10^-} 3-|x-10| = \lim_{x\to 10^-} 3-(-x+10) = \lim_{x\to 10^-} x-7 = 10-7 = 3$$

Thus

$$\lim_{x\to 10} 3-|x-10| = 3$$

Example (Using Squeeze Theorem *(continued)*)

- **Claim C:** $\lim_{x\to 10} 3 + |x-10| = 3$
- $$\lim_{x\to 10^+} 3 + |x-10| = \lim_{x\to 10^+} 3 + (x-10) = \lim_{x\to 10^+} x - 7 = 10 - 7 = 3$$
  $$\lim_{x\to 10^-} 3 + |x-10| = \lim_{x\to 10^-} 3 + (-x+10) = \lim_{x\to 10^-} -x + 13 = -10 - 13 = 3$$
  Thus
  $$\lim_{x\to 10} 3 + |x-10| = 3$$
- Now combining Claims A, B and C and applying the Squeeze Theorem we can conclude that
  $$\lim_{x\to 10} g(x) = 3.$$

Infinite Limits

Infinite limits

If $f(x)$ gets arbitrarily large as x gets close to but not equal to a then we say

$$\lim_{x\to a} f(x) = \infty$$

Note: If $\lim_{x\to a} f(x) = \infty$ then $\lim_{x\to a} f(x)$ **does not exist**. Saying $\lim_{x\to a} f(x) = \infty$ gives us info on exactly why the limit does not exist.

Example (Infinite limits from graphs)

CAREFUL GRAPHS

Definition (Vertical Asymptote)

If $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a^+} f(x) = \pm\infty$, or $\lim_{x\to a^-} f(x) = \pm\infty$ then $x = a$ is a **vertical asymptote** of f.

Infinite Limits

Example (Infinite Limit)

Compute $\lim_{x\to -2^+} \frac{x^2}{x^2-4}$

- STEP I: Factor

$$\frac{x^2}{x^2-4} = \frac{x^2}{(x+2)(x-2)}$$

- NOTE: 2 is a root of denominator and NOT numerator so expect infinite limit
- Q: Is limit $+\infty$ or $-\infty$? If $x > -2$ and x is close to -2 then $x < 0$, $(x+2) > 0$, and $(x-2) < 0$ so $\frac{x^2}{(x+2)(x-2)} > 0$
-
$$\lim_{x\to -2^+} \frac{x^2}{x^2-4} = -\infty$$

Infinite Limits

Example (Vertical asymptotes)

Find all vertical asymptotes of $h(t) = \frac{t^2-3t-4}{t^2-6t-7}$

- Factor

$$\frac{t^2-3t-4}{t^2-6t-7} = \frac{(t-4)(t+1)}{(t-7)(t+1)}$$

- $\lim_{t\to 7^+} \frac{(t-4)(t+1)}{(t-7)(t+1)} = \infty$
- $\lim_{t\to 7^-} \frac{(t-4)(t+1)}{(t-7)(t+1)} = -\infty$
- $\lim_{t\to -1} \frac{(t-4)(t+1)}{(t-7)(t+1)} = \lim_{t\to -1} \frac{t-4}{t-7} = \frac{-1-4}{-1-7} = \frac{5}{8}$ so $t = -1$ is **not** a vertical asymptote
- $\boxed{h \text{ has a single vertical asymptote at } t = 7}$

Infinite Limits

Example (Finite and Infinite limits)

Compute the following limits.

- $$\lim_{x\to 4^+} \frac{x^2+4x-5}{x^2-6x+8} = \lim_{x\to 4^+} \frac{(x-5)(x+1)}{(x-2)(x-4)} = -\infty$$

- $$\begin{aligned} \lim_{x\to -2^-} \frac{x+2}{x^2+2x} &= \lim_{x\to -2^-} \frac{x+2}{x(x+2)} \\ &= \lim_{x\to -2^-} \begin{cases} \frac{1}{x}, & x\neq -2 \\ \text{undefined}, & x=-2 \end{cases} \\ &= \lim_{x\to -2^-} \tfrac{1}{x} \\ &= -\tfrac{1}{2} \end{aligned}$$

Vertical Asymptotes

Example (Vertical Asymptotes)

Find all of the vertical asymptotes of the form $x = a$ of the following

- $$f(x) = \frac{x^2-9}{x^2+5x+6} = \frac{(x+3)(x-3)}{(x+2)(x+3)} = \begin{cases} \frac{x-3}{x+2}, & x\neq -3 \\ \text{undefined}, & x=-3 \end{cases}$$

  f has a single vertical asymptote at $x = -2$

- $$g(x) = \frac{x}{(x^2-4)^6} = \frac{x}{(x-2)^6(x+2)^6}$$

  g has a two vertical asymptotes at $x = -2$ and $x = 2$