Lecture 6 – September 5, 2018

- Office hours beginning today – M & W 12:30pm-2pm in MW 650
- Quiz 2 – Thursday 9/6 in recitation

Today

- Precise definition of limit

Greek Alphabet

- Math needs lots of variables
- Computer Science solution: use words as variables (Good)
- Math solution: co-opt other alphabets (Mostly Greek) (Bad, but it's tradition)

Greek Alphabet

alpha, α, A	beta, β, B	gamma, γ, Γ	delta, δ, Δ
epsilon, ε, E	zeta, ζ, Z	eta, η, H	theta, θ, Θ
iota, ι, I	kappa, κ, K	lambda, λ, Λ	mu, μ, M
nu, ν, N	xi, ξ, Ξ	omicron, o, O	pi, π, Π
rho, ρ, P	sigma, σ, Σ	tau, τ, T	upsilon, υ, Y
phi, φ, Φ	chi, χ, X	psi, ψ, Ψ	omega, ω, Ω

Definition of Limit

Definition (Limit)

$$\lim_{x \to a} f(x) = L$$

if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies that $|f(x) - L| < \varepsilon$.

- Picture
- Circus cannon

Limit Proofs

Example (Limit Proof from definition)

Using only the definition of the limit show that $\lim_{x\to 2}(3x-5) = 1$.

- Given $\varepsilon > 0$ choose $\delta = \underline{\;\frac{\varepsilon}{3}\;}$.
- Then if $0 < |x-2| < \delta$ it follows that

$$\begin{aligned} |f(x) - L| &= |(3x-5) - 1| \\ &= |3x - 6| \\ &= 3|x-2| \\ &< 3\delta \\ &< 3 \cdot \tfrac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

Limit Proofs

Illegal choice of δ

Using only the definition of the limit show that $\lim_{x\to -3} x^2 = 9$.

- Given $\varepsilon > 0$ choose $\delta = \underline{\;\frac{\varepsilon}{|x-3|}\;}$. **Not allowed!!! Only ε can be used in choice of δ**
- Then if $0 < |x-(-3)| < \delta$ it follows that

$$\begin{aligned} |f(x) - L| &= |x^2 - 9| \\ &= |(x-3)(x+3)| \\ &= |x-3||x+3| \\ &< |x-3|\delta \\ &< |x-3| \cdot \tfrac{\varepsilon}{|x-3|} \\ &= \varepsilon \end{aligned}$$

- **Invalid proof!!!**

Limit Proofs

Example (Correct version of proof above)

Using only the definition of the limit show that $\lim_{x\to -3} x^2 = 9$.

- Given $\varepsilon > 0$ choose $\delta = \underline{\quad \min\left\{\frac{\varepsilon}{7}, 1\right\} \quad}$.
- Then if $0 < |x+3| < \delta$ it follows that

$$
\begin{aligned}
-\delta &< x+3 < \delta \\
-1 &< x+3 < 1 \\
-1-6 &< x+3-6 < 1-6 \\
-7 &< x-3 < -5 \\
5 &< |x-3| < 7
\end{aligned}
$$

Example (Correct version of proof above *(continued)*)

- Hence if $0 < |x+3| < \delta$ it follows that

$$
\begin{aligned}
|f(x) - L| &= |x^2 - 9| \\
&= |(x-3)(x+3)| \\
&= |x-3||x+3| \\
&< |x-3|\delta \\
&< 7 \cdot \frac{\varepsilon}{7} \\
&= \varepsilon
\end{aligned}
$$

□

Definition of Limit

Definition (L is not the limit)

$$\lim_{x \to a} f(x) \neq L$$

if for some $\varepsilon > 0$ it is true that for all $\delta > 0$ there is some x such that $0 < |x - a| < \delta$ but $|f(x) - L| \geq \varepsilon$.

Limit Proofs

Example (4 is not the limit)

Show that $\lim_{x \to 1} x + 2 \neq 4$ using only the definition of the limit.

- Choose $\varepsilon = \frac{1}{2}$ and suppose $\delta > 0$.
- Choose $x = 1 + \min\left\{\frac{1}{2}, \frac{\delta}{2}\right\}$. Then

$$|x - 1| = |1 + \min\left\{\tfrac{1}{2}, \tfrac{\delta}{2}\right\} - 1| = \min\left\{\tfrac{1}{2}, \tfrac{\delta}{2}\right\} < \delta$$

but

$$\begin{aligned} |(x + 2) - 4| &= |1 + \min\left\{\tfrac{1}{2}, \tfrac{\delta}{2}\right\} + 2 - 4| \\ &= |-1 + \min\left\{\tfrac{1}{2}, \tfrac{\delta}{2}\right\}| \\ &= 1 - \min\left\{\tfrac{1}{2}, \tfrac{\delta}{2}\right\} \\ &\geq \tfrac{1}{2} \\ &= \varepsilon \end{aligned}$$

□