Lecture 7 – September 7, 2018

- Office hours – M & W 12:30pm-2pm in MW 650

Today

- Definition of the derivative
- Nondifferentiability

Definition of Limit

Definition (L is not the limit)

$$\lim_{x \to a} f(x) \neq L$$

if for some $\varepsilon > 0$ it is true that for all $\delta > 0$ there is some x such that $0 < |x - a| < \delta$ but $|f(x) - L| \geq \varepsilon$.

Limit Proofs

Example (4 is not the limit)

Show that $\lim_{x \to 1} x + 2 \neq 4$ using only the definition of the limit.

- Choose $\varepsilon = \frac{1}{2}$ and suppose $\delta > 0$.
- Choose $x = 1 + \min\left\{\frac{1}{2}, \frac{\delta}{2}\right\}$. Then

$$|x - 1| = \left|1 + \min\left\{\frac{1}{2}, \frac{\delta}{2}\right\} - 1\right| = \min\left\{\frac{1}{2}, \frac{\delta}{2}\right\} < \delta$$

but

$$\begin{aligned} |(x + 2) - 4| &= \left|1 + \min\left\{\tfrac{1}{2}, \tfrac{\delta}{2}\right\} + 2 - 4\right| \\ &= \left| -1 + \min\left\{\tfrac{1}{2}, \tfrac{\delta}{2}\right\} \right| \\ &= 1 - \min\left\{\tfrac{1}{2}, \tfrac{\delta}{2}\right\} \\ &\geq \tfrac{1}{2} \\ &= \varepsilon \end{aligned}$$

□

Definition of the derivative at a point

Definition (Derivative at a point)

If $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists then f is **differentiable at** a and we define the **derivative of** f **at** a to be

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$f'(a)$ is also called the **instantaneous rate of change** of f at a and is the **slope of the tangent line** to f at the point $(a,(f(a))$.
If $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ does not exist then f is **not differentiable** at a

Definition (Equivalent definition of the derivative)

The **derivative of** f **at** a is

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

Derivatives

Example (Derivative from definition)

Use the definition of the derivative to compute $f'(2)$ if $f(x) = 3x^2$

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\
&= \lim_{h\to 0} \frac{3(2+h)^2 - 3\cdot 2^2}{h} \\
&= \lim_{h\to 0} \frac{3(4+4h+h^2)-12}{h} \\
&= \lim_{h\to 0} \frac{12+12h+3h^2-12}{h} \\
&= \lim_{h\to 0} \frac{12h+3h^2}{h} \\
&= \lim_{h\to 0} 12+3h \\
&= 12+3\cdot 0 \\
&= 12
\end{aligned}$$

Derivative as a function

Definition (Derivative function)

The **derivative** of f is the *function*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

or equivalently

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}.$$

The domain of f' is the set of all $x \in \mathbf{R}$ where f is differentiable.

Derivatives

Example (Derivative from definition)

Use the definition to compute $g'(x)$ if $g(x) = \frac{2}{x-1}$

$$\begin{aligned} g'(x) &= \lim_{t \to x} \frac{g(t) - g(x)}{t - x} \\ &= \lim_{t \to x} \frac{\frac{2}{t-1} - \frac{2}{x-1}}{t - x} \\ &= \lim_{t \to x} \frac{\frac{2(x-1)}{(x-1)(t-1)} - \frac{2(t-1)}{(x-1)(t-1)}}{t - x} \\ &= \lim_{t \to x} \frac{\frac{2x-2}{(x-1)(t-1)} - \frac{2t-2}{(x-1)(t-1)}}{t - x} \\ &= \lim_{t \to x} \frac{\left(\frac{2x-2-2t+2}{(x-1)(t-1)}\right)}{t - x} \end{aligned}$$

Example (Derivative from definition *(continued)*)

$$
\begin{aligned}
&= \lim_{t \to x} \frac{\left(\frac{-2(t-x)}{(x-1)(t-1)} \right)}{t - x} \\
&= \lim_{t \to x} \frac{-2}{(x-1)(t-1)} \\
&= \frac{-2}{(x-1)(x-1)} \\
&= \boxed{\tfrac{-2}{(x-1)^2}}
\end{aligned}
$$

Tangent Lines

Example (Tangent lines)

Find the slope of the tangent line to $j(x) = \sqrt{2x}$ at the point $P = (1, \sqrt{2})$

Note: We have a point on tangent line so we just need slope of tangent line.

$$
\begin{aligned}
j'(1) &= \lim_{h \to 0} \frac{j(1+h) - j(1)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{2(1+h)} - \sqrt{2 \cdot 1}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{2+2h} - \sqrt{2}}{h} \cdot \frac{\sqrt{2+2h} + \sqrt{2}}{\sqrt{2+2h} + \sqrt{2}} \\
&= \lim_{h \to 0} \frac{(\sqrt{2+2h})^2 - (\sqrt{2})^2}{h(\sqrt{2+2h} + \sqrt{2})} \\
&= \lim_{h \to 0} \frac{2 + 2h - 2}{h(\sqrt{2+2h} + \sqrt{2})}
\end{aligned}
$$

Example (Tangent lines *(continued)*)

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{2h}{h(\sqrt{2+2h}+\sqrt{2})} \\
&= \lim_{h\to 0} \frac{2}{(\sqrt{2+2h}+\sqrt{2})} \\
&= \frac{2}{(\sqrt{2+2\cdot 0}+\sqrt{2})} \\
&= \frac{1}{\sqrt{2}}
\end{aligned}
$$

The tangent line to $j(x) = \sqrt{2x}$ at the point $P = (1, \sqrt{2})$ has equation:

$$\boxed{y = \tfrac{1}{\sqrt{2}}(x-1)+\sqrt{2}}$$

Notation

Notation

Lagrange	Leibniz	Leibniz	Euler
f'	$\frac{\mathrm{d}}{\mathrm{d}\,x} f$	$\frac{\mathrm{d}\,f}{\mathrm{d}\,x}$	$D_x f$
$f'(a)$	$\frac{\mathrm{d}}{\mathrm{d}\,x} f\Big\vert_{x=a}$	$\frac{\mathrm{d}\,f}{\mathrm{d}\,x}\Big\vert_{x=a}$	$D_x f\big\vert_{x=a}$

Derivatives

Theorem (Differentiable implies continuous)

If f is differentiable at a then f is continuous at a.

Example (Contrapositive)

- Following are logically equivalent statements (if one is true then so is the other):
- If x is a cat then x is a mammal.
- **CONTRAPOSITVE:** If x is not a mammal then x is not a cat.
- **CONVERSE:** If x is a mammal then x is a cat.
- What is contrapositive of theorem above?

Corollary (Not continuous implies not differentiable)

CONTRAPOSITVE: *If f is not continuous at a then f is not differentiable at a.*

Nondifferentiability

Example (Nondifferentiability)

- The following functions are not differentiable at 2
- ONLY DISCONTINUOUS EXAMPLES

Nondifferentiability

- Recall $g(t) = |t|$ is not differentiable at $t = 0$ since $\lim_{t\to 0^-} \frac{|0+h|-|0|}{h} = -1$ and $\lim_{t\to 0^+} \frac{|0+h|-|0|}{h} = 1$ so $\lim_{t\to 0} \frac{|0+h|-|0|}{h}$ DNE
- Qualitatively we say g has a **corner** at $t = 0$

Example (Nondifferentiability)

- The following functions are not differentiable at -3
- ONLY CORNER EXAMPLES

Nondifferentiability

Example (Vertical tangent)

Use the definition of the derivative to compute $f'(0)$ if

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ -\sqrt{-x}, & x < 0 \end{cases}$$

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{\sqrt{0+h}-\sqrt{0}}{h} = \lim_{h\to 0^+} \frac{\sqrt{h}}{h} = \lim_{h\to 0^+} \frac{1}{\sqrt{h}} = \infty$$

$$\begin{aligned} \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^-} \frac{-\sqrt{-(0+h)}-\sqrt{0}}{h} \\ &= \lim_{h\to 0^-} \frac{\sqrt{-h}}{-h} = \lim_{h\to 0^-} \frac{\sqrt{-h}}{\sqrt{-h}\cdot\sqrt{-h}} \\ &= \lim_{h\to 0^-} \frac{1}{\sqrt{-h}} \\ &= \infty \end{aligned}$$

Nondifferentiability

Example (Nondifferentiability)

- The following functions are not differentiable at 5
- ONLY VERTICAL TANGENT EXAMPLES
- NOTE: Still have a tangent line (it's vertical)

In summary a function f is not differentiable at a if **any** of the following are true:

1. f is not continuous at a
2. f has a corner at a
3. f has a vertical tangent line at $(a, f(a))$

Questions to think about

Questions

1. How do the domains of $g(x) = |x|$ and g' compare?
2. How should the domains of f and f' compare in general?
3. Can f be differentiable at a if $lim_{x \to a} f(x) = \infty$?
4. What limits will we need to know to use the definition of the derivative to compute $f'(x)$ if $f(x) = \sin x, \cos x, e^x$?