Lecture 8 – September 10, 2018

- Office hours – M & W 12:30pm-2pm in MW 650

Today

- Derivative Rules
- Atomic Derivatives
- Higher Derivatives

What Derivative Rules Do We Need?

- We've discussed atomic limits and limit laws
- We've discussed atomic continuous functions and continuity laws
- What derivative rules would be helpful for us?
- What atomic derivatives will we need?

Helpful Derivative Rules

1. Sum Rule $\frac{d}{dx}(f(x) + g(x)) =?$
2. Product Rule $\frac{d}{dx}(f(x) \cdot g(x)) =?$
3. Quotient Rule $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) =?$
4. Chain Rule $\frac{d}{dx}(f(g(x)) =?$
5. Inverse function Rule $\frac{d}{dx}f^{-1}(x) =?$

What Atomic Derivatives Do We Need?

- Now assume we have Derivative Rules above.
- What atomic derivatives do we need?

Necessary Atomic Derivatives

1. $\frac{d}{dx}c =?$
2. $\frac{d}{dx}x =?$
3. $\frac{d}{dx}e^x =?$
4. $\frac{d}{dx}\sin x =?$
5. $\frac{d}{dx}\cos x =?$

What About the “Power Rule?”

Example

Given Derivative Rules and Atomic Derivatives how would you compute the following:

- $\frac{d}{dx}x^2$? – Product Rule & $\frac{d}{dx}x$
- $\frac{d}{dx}x^{-1}$? – Quotient Rule & $\frac{d}{dx}1$ & $\frac{d}{dx}x$
- $\frac{d}{dx}x^3$? – Product Rule & $\frac{d}{dx}x$ & $\frac{d}{dx}x^2$
- $\frac{d}{dx}\ln x =$? – Inverse function Rule & $\frac{d}{dx}e^x =$
- $\frac{d}{dx}\cot x =$? – Quotient Rule & $\frac{d}{dx}\sin x$ & $\frac{d}{dx}\cos x$
- $\frac{d}{dx}\arctan x =$? – Quotient Rule & $\frac{d}{dx}\sin x$ & $\frac{d}{dx}\cos x$ (to get $\frac{d}{dx}\tan x$) then Inverse function Rule
- $\frac{d}{dx}5^x$? – $5^x = e^{(x \ln 5)}$ Chain Rule & $\frac{d}{dx}e^x$ & Product Rule & $\frac{d}{dx}x$ & $\frac{d}{dx}c$
- $\frac{d}{dx}\sqrt[3]{x}$? – Two approaches: (1) $\sqrt[3]{x}$ is inverse function for x^3 OR (2) $\sqrt[3]{x} = x^{\frac{1}{3}} = e^{(\frac{1}{3}\ln x)}$

Some Atomic Derivatives

Theorem (Some Atomic Derivatives)

1. $\frac{d}{dx}c = 0$
2. $\frac{d}{dx}x = 1$

Proof.

$$\frac{d}{dx}c = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0}\frac{0}{h} = \lim_{h\to 0} 0 = 0$$

$$\frac{d}{dx}x = \lim_{h\to 0}\frac{(x+h)-x}{h} = \lim_{h\to 0}\frac{h}{h} = \lim_{h\to 0} 1 = 1$$

□

Another Atomic Derivative

A fundamental property of e^x

For all a and b

$$e^{a+b} = e^a e^b$$

Theorem (Another Atomic Derivative)

3. $\frac{d}{dx} e^x = e^x$

Proof.

$$\begin{aligned}
\frac{d}{dx} e^x &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x e^h - e^x}{h} \\
&= \lim_{h \to 0} e^x \cdot \frac{e^h - 1}{h} \\
&= e^x \cdot \lim_{h \to 0} \frac{e^h - 1}{h}
\end{aligned}$$

□

Definition (The number e)

e is the unique real number satisfying:

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

Sum Rule

Theorem (Sum Rule)

If f and g are differentiable at x then $f+g$ is differentiable at x and

$$\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x).$$

Proof.

Suppose f and g are differentiable at x.

$$\begin{aligned}\frac{d}{dx}(f(x)+g(x)) &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-(f(x)+g(x))}{h}\\ &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right)\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)+g'(x)\end{aligned}$$

□

A Useful Derivative

Theorem (Power Rule *(not atomic but useful!!!)*)

If $n \in \mathbf{N}$ then

$$\frac{d}{dx}x^n = nx^{n-1}$$

Proof.

$$\begin{aligned}\left.\frac{d}{dx}x^n\right|_{x=a} &= \lim_{x\to a}\frac{x^n-a^n}{x-a}\\ &= \lim_{x\to a}\frac{(x-a)(x^{n-1}+x^{n-2}a+\cdots+xa^{n-2}+a^{n-1})}{x-a}\\ &= \lim_{x\to a}x^{n-1}+x^{n-2}a+\cdots+xa^{n-2}+a^{n-1}\\ &= a^{n-1}+a^{n-2}a+\cdots+aa^{n-2}+a^{n-1}\\ &= na^{n-1}\end{aligned}$$

□

Using Power Rule

Example (Using Power Rule)

Use the Power Rule to compute the following derivatives.

- $\frac{\mathrm{d}}{\mathrm{d}\,x}x^{10} = 10x^9$
- $\frac{\mathrm{d}}{\mathrm{d}\,x}x^{43} = 43x^{42}$
- $\frac{\mathrm{d}}{\mathrm{d}\,x}x = 1x^0 = 1$

Constant Multiple Rule

Theorem (Constant Multiple Rule *(redundant given power rule but useful!!!)*)

If f is differentiable at x and c is a constant then cf is differentiable at x and

$$(cf)'(x) = cf'(x).$$

Proof.

Can you prove this? □

Derivatives using derivative rules

Example (Derivatives using derivative rules)

Compute the following derivatives using the Sum Rule, Power Rule, Constant Multiple Rule and known Atomic Derivatives

- $\frac{d}{dx} 6x^3 = 6 \cdot 3x^2 = 18x^2$
- $\frac{d}{dx} \frac{x}{8} = \frac{1}{8} \cdot 1x^0 = \frac{1}{8}$
- $\frac{d}{dx} e^3 x^4 = 4e^3 x^3$
- $\frac{d}{dx} \left(6x^5 + \sin(23)x^4 + \pi x^3 + \sqrt{3}x\right) = 30x^4 + 4\sin(23)x^3 + 3\pi x^2 + \sqrt{3}$

Higher derivatives

Definition (Higher derivatives)

$$f^{(0)}(x) = f(x).$$

If $n \in \mathbf{N}$ then the nth derivative of f is

$$f^{(n)}(x) = \frac{d}{dx} f^{(n-1)}(x).$$

Notation

$$\frac{d^n}{dx^n} f(x) = f^{(n)}(x)$$
$$f''(x) = f^{(2)}(x)$$
$$f'''(x) = f^{(3)}(x)$$

Higher derivatives

Example (Higher derivatives)

$$\begin{aligned}
\frac{d^3}{dx^3}\left(6x^5+4e^x+4x^2\right) &= \frac{d}{dx}\left(\frac{d^2}{dx^2}\left(6x^5+4e^x+4x^2\right)\right)\\
&= \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(6x^5+4e^x+4x^2\right)\right)\right)\\
&= \frac{d}{dx}\left(\frac{d}{dx}\left(30x^4+4e^x+8x\right)\right)\\
&= \frac{d}{dx}\left(120x^3+4e^x+8\right)\\
&= 360x^3+4e^x
\end{aligned}$$

Questions to think about

Questions

1. What is $\frac{d^8}{dx^8}x^3$?
2. What is $\frac{d^8}{dx^8}e^x$?
3. What is $\frac{d^8}{dx^8}x^8$?