Lecture 9 – September 12, 2018

- Office hours – M & W 12:30pm-2pm in MW 650
- Midterm 1 either Thurs. 9/20 at 6pm (in room TBD) OR Fri. 9/21 during class
- Practice for Midterm 1 posted today
- Written Homework 2 due in recitation on Tues. 9/18

Today

- Product Rule
- Quotient Rule
- Trig Limits
- Trig Derivatives

Derivative Rules

Derivative Rules

1. Sum Rule $\frac{\mathrm{d}}{\mathrm{d}\,x}(f(x)+g(x)) = f'(x)+g'(x)$
2. Product Rule $\frac{\mathrm{d}}{\mathrm{d}\,x}(f(x)\cdot g(x)) = f'(x)g(x)+f(x)g'(x)$
3. Quotient Rule $\frac{\mathrm{d}}{\mathrm{d}\,x}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$
4. Chain Rule $\frac{\mathrm{d}}{\mathrm{d}\,x}(f(g(x)) = ?$
5. Inverse function Rule $\frac{\mathrm{d}}{\mathrm{d}\,x}f^{-1}(x) = ?$

Atomic Derivatives

Necessary Atomic Derivatives

1. $\frac{\mathrm{d}}{\mathrm{d}\,x}c = 0$
2. $\frac{\mathrm{d}}{\mathrm{d}\,x}x = 1$
3. $\frac{\mathrm{d}}{\mathrm{d}\,x}e^x = e^x$
4. $\frac{\mathrm{d}}{\mathrm{d}\,x}\sin x = \cos x$
5. $\frac{\mathrm{d}}{\mathrm{d}\,x}\cos x = -\sin x$

Proof of Product Rule

Theorem (Product Rule)

If f and g are differentiable at x then

$$\frac{\mathrm{d}}{\mathrm{d}\,x}(f(x)\cdot g(x)) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

Proof.

Suppose f and g are differentiable at x. Then

$$\tfrac{\mathrm{d}}{\mathrm{d}\,x}(f(x)\cdot g(x))$$

$$\begin{aligned}
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}
\end{aligned}$$

Proof *(continued)*.

$$\begin{aligned}
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h\to 0} g(x+h) + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x)\cdot g(x) + f(x)g'(x)
\end{aligned}$$

$\square$

Proof of Quotient Rule

Theorem (Quotient Rule)

If f and g are differentiable at x and $g(x) \neq 0$ then

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

Proof.

Suppose g differentiable at x and $g(x) \neq 0$. Then

$$\begin{aligned}
\frac{d}{dx}\frac{1}{g(x)} &= \lim_{h\to 0}\frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} \\
&= \lim_{h\to 0}\frac{\left(\frac{g(x)-g(x+h)}{g(x)g(x+h)}\right)}{h} \\
&= \lim_{h\to 0}\left(-\frac{g(x+h)-g(x)}{h}\right)\left(\frac{1}{g(x)}\right)\left(\frac{1}{g(x+h)}\right) \\
&= -\frac{g'(x)}{g^2(x)}
\end{aligned}$$

Proof *(continued)*.

Suppose further that f is differentiable at x. Then

$$\begin{aligned}
\frac{d}{dx}\frac{f(x)}{g(x)} &= \frac{d}{dx}\left(f(x)\cdot\frac{1}{g(x)}\right) \\
&= f'(x)\frac{1}{g(x)} + f(x)\left(-\frac{g'(x)}{g^2(x)}\right) \\
&= f'(x)\frac{g(x)}{g^2(x)} + f(x)\left(-\frac{g'(x)}{g^2(x)}\right) \\
&= \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{[g(x)]^2}
\end{aligned}$$

□

Using the Product and Quotient Rules

Example (Using the Product and Quotient Rules)

1. $\frac{\mathrm{d}}{\mathrm{d}\,x}(x^2e^x) = 2xe^x + x^2e^x$
2. $\frac{\mathrm{d}}{\mathrm{d}\,x}(e^{2x}) = \frac{\mathrm{d}}{\mathrm{d}\,x}(e^xe^x) = e^xe^x + e^xe^x = 2e^{2x}$
3. $\frac{\mathrm{d}}{\mathrm{d}\,x}(x^3x^8) = \frac{\mathrm{d}}{\mathrm{d}\,x}(x^{11}) = 11x^{10}$ but also
   $\frac{\mathrm{d}}{\mathrm{d}\,x}(x^3x^8) = 3x^2x^8 + x^38x^7 = 11x^{10}$
4. $\frac{\mathrm{d}}{\mathrm{d}\,x}(\frac{x^2}{e^x}) = \frac{2xe^x - x^2e^x}{[e^x]^2} = \frac{2x - x^2}{e^x}$
5. $\frac{\mathrm{d}}{\mathrm{d}\,x}(\frac{2x}{x^2+1}) = \frac{2(x^2+1) - 2x2x}{(x^2+1)^2} = \frac{-2(x^2-1)}{(x^2+1)^2}$

Variables are blanks

Variables are blanks. If $f(x) = \sin x + \cos x$ what is:

1. $f(a)$? $f(a) = \sin a + \cos a$
2. what is $f(10)$? $f(10) = \sin 10 + \cos 10$
3. What is $f(a+h)$? $f(a+h) = \sin(a+h) + \cos(a+h)$
4. What is $f(f(x))$?

$$\begin{aligned} f(f(x)) &= \sin(f(x)) + \cos(f(x)) \\ &= \sin(\sin x + \cos x) + \cos(\sin x + \cos x) \end{aligned}$$

5. Why not use blanks?

Trig Facts

1. $\sin(a+b) = \sin a \cos b + \cos a \sin b$
2. $\cos(a+b) = \cos a \cos b - \sin a \sin b$
3. $\lim_{h\to 0} \frac{\sin h}{h} = 1$
4. $\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$

Define:

$$S(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

$$C(x) = \begin{cases} \frac{\cos x - 1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

What are the largest intervals of continuity for S and C? (Answer: Both are continuous on $(-\infty, \infty)$)
Recall:

Theorem

If $\lim_{x\to a} g(x)$ *exists and* f *is continuous at* $\lim_{x\to a} g(x)$ *then*

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

Example (Trig Limits)

Compute $\lim_{x\to 0} \frac{\sin 2x}{x}$ using the definition of the the derivative.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 2x}{x} &= \lim_{x\to 0} 2 \cdot \frac{\sin 2x}{2x} \\
&= \lim_{x\to 0} 2 \cdot S(2x) \\
&= 2 \lim_{x\to 0} S(2x) \\
&= 2S\left(\lim_{x\to 0} 2x\right) \\
&= 2 \cdot S(0) \\
&= 2 \cdot 1 \\
&= 2
\end{aligned}$$

Example (Trig Limits)

Compute $\lim_{x\to 1} \frac{\cos 4 \ln x - 1}{\ln x}$ using the definition of the the derivative.

$$\begin{aligned}
\lim_{x\to 1} \frac{\cos 4 \ln x - 1}{\ln x} &= \lim_{x\to 1} 4 \cdot \frac{\cos 4 \ln x - 1}{4 \ln x} \\
&= \lim_{x\to 1} 4 \cdot C(4 \ln x) \\
&= 4 \lim_{x\to 1} C(4 \ln x) \\
&= 4C(\lim_{x\to 1} 4 \ln x) \\
&= 4 \cdot C(0) \\
&= 4 \cdot 0 \\
&= 0
\end{aligned}$$

Theorem (Derivative of $\sin x$ and $\cos x$)

4. $\frac{d}{dx}\sin x = \cos x$
5. $\frac{d}{dx}\cos x = -\sin x$

Proof.

$$\begin{aligned}
\tfrac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h-\sin x}{h}+\lim_{h\to 0}\frac{\cos x\sin h}{h}\\
&= \sin x\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h}\\
&= \sin x\cdot 0+\cos x\cdot 1\\
&= \cos x
\end{aligned}$$

Try proving $\frac{d}{dx}\cos x = -\sin x$ yourself. □

Example (Derivative of tan)

Other trig derivatives are now exercises:

$$\begin{aligned}
\tfrac{d}{dx}\tan x &= \tfrac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\
&= \frac{\cos x\cos x-\sin x(-\sin x)}{\cos^2 x}\\
&= \frac{\cos^2 x+\sin^2 x}{\cos^2 x}\\
&= \frac{1}{\cos^2 x}\\
&= \sec^2 x
\end{aligned}$$

Example (Derivative of csc)

$$\begin{aligned}\tfrac{d}{dx}\csc x &= \tfrac{d}{dx}\left(\frac{1}{\sin x}\right)\\ &= \frac{0\cdot\sin x - 1\cdot\cos x}{\sin^2 x}\\ &= -\frac{\cos x}{\sin^2 x}\\ &= -\frac{\cos x}{\sin x}\cdot\frac{1}{\sin x}\\ &= -\cot x\csc x\end{aligned}$$

Trig derivatives

$\tfrac{d}{dx}\sin x = \cos x$
$\tfrac{d}{dx}\cos x = -\sin x$
$\tfrac{d}{dx}\tan x = \sec^2 x$
$\tfrac{d}{dx}\cot x = -\csc^2 x$
$\tfrac{d}{dx}\sec x = \sec x\tan x$
$\tfrac{d}{dx}\csc x = -\csc x\cot x$

More Derivatives Required for Homework

The following derivatives should be computed using the **Chain Rule** or **Inverse Function Rule**, but might be needed for homework. They can also be proven directly from definition of the derivative (can you do this?)

$\frac{\mathrm{d}}{\mathrm{d}\,x}\sqrt{x} = \frac{1}{2\sqrt{x}}$
$\frac{\mathrm{d}}{\mathrm{d}\,x}e^{kx} = ke^{kx}$ for any $k \in \mathbf{R}$

Example (More Derivative Computations)

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}\sec t &= \frac{\mathrm{d}}{\mathrm{d}\,x}(\sec t \tan t) \\ &= (\sec t \tan t)\cdot \tan t + \sec t \sec^2 t \\ &= \sec t \tan^2 t + \sec^3 t \\ &= \sec t(\sec^2 t - 1) + \sec^3 t \\ &= 2\sec^3 t - \sec t\end{aligned}$$

Example (More Derivative Computations)

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left(\frac{e^x}{\sin x + \cos x}\right) = ???$$