Lecture 13 – September 24, 2018

- My office hours – M & W 12:30pm-2pm in Math Tower (MW) 650
- MSLC – `https://mslc.osu.edu/courses/math/1161`

Today

- Derivatives of inverse functions
- Derivative of $\ln x$
- Derivative of inverse trig functions
- Logarithmic differentiation

Derivative Rules

All Derivative Rules

1. Sum Rule $\frac{\mathrm{d}}{\mathrm{d}x}(f(x)+g(x)) = f'(x)+g'(x)$
2. Product Rule $\frac{\mathrm{d}}{\mathrm{d}x}(f(x)\cdot g(x)) = f'(x)g(x)+f(x)g'(x)$
3. Quotient Rule $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$
4. Chain Rule $\frac{\mathrm{d}}{\mathrm{d}x}(f(g(x)) = f'(g(x))g'(x)$
5. Inverse function Rule $\frac{\mathrm{d}}{\mathrm{d}x}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$

Atomic Derivatives

All Atomic Derivatives

1. $\frac{\mathrm{d}}{\mathrm{d}x}c = 0$
2. $\frac{\mathrm{d}}{\mathrm{d}x}x = 1$
3. $\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$
4. $\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$
5. $\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$

Derivative of inverse functions

Theorem (Derivative of an inverse function)

Suppose $f(a) = b$ and f is differentiable at a. If $f'(a) \neq 0$ then f^{-1} is differentiable at b and

$$\frac{d}{dx} f^{-1}(x)\Big|_{x=b} = \frac{1}{f'(f^{-1}(b))}$$

Partial Proof.

We will assume that f^{-1} is differentiable without proof.

$$\begin{aligned} f(f^{-1}(x)) &= x \\ \frac{d}{dx} f(f^{-1}(x)) &= \frac{d}{dx} x \\ f'(f^{-1}(x)) \frac{d}{dx} f^{-1}(x) &= 1 \\ \frac{d}{dx} f^{-1}(x) &= \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

□

Derivative of ln

Derivative of ln

Solution I

$$e^{\ln x} = x \quad \text{for } x > 0$$

$$\begin{aligned} \frac{d}{dx}(e^{\ln x}) &= \frac{d}{dx} x \\ e^{\ln x} \frac{d}{dx} \ln x &= 1 \\ \frac{d}{dx} \ln x &= \frac{1}{e^{\ln x}} \\ \frac{d}{dx} \ln x &= \frac{1}{x} \end{aligned}$$

Solution II

$$\begin{aligned} \frac{d}{dx} \ln x &= \frac{1}{e^{\ln x}} \\ \frac{d}{dx} \ln x &= \frac{1}{x} \end{aligned}$$

$$\boxed{\frac{d}{dx} \ln x = \frac{1}{x} \text{ for } x > 0}$$

Derivative of inverse trig functions

Derivative of arcsin x

Let $f(x) = \sin x$ and $f^{-1}(x) = \arcsin x$.

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\cos(\arcsin x)}$$
$$= \frac{1}{\sqrt{1 - x^2}}$$

$$\boxed{\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \text{ for } -1 < x < 1}$$

Derivative of inverse trig functions

Derivative of arctan x

$$\tan(\arctan x) = x \qquad \text{for all } x \in \mathbf{R}$$
$$\frac{d}{dx}(\tan(\arctan x)) = \frac{d}{dx}x$$
$$\sec^2(\arctan x)\frac{d}{dx} \arctan x = 1$$
$$\frac{d}{dx} \arctan x = \frac{1}{\sec^2(\arctan x)}$$
$$\frac{d}{dx} \arctan x = \cos^2(\arctan x)$$
$$\frac{d}{dx} \arctan x = \frac{1}{1 + x^2}$$

$$\boxed{\frac{d}{dx} \arctan x = \frac{1}{1+x^2} \text{ for all } x \in \mathbf{R}}$$

Derivative of inverse trig functions

Derivative of $\sec^{-1} x$

$$\begin{aligned}
\sec(\sec^{-1} x) &= x \quad \text{for all } x \in (-\infty, -1) \cup (1, \infty) \\
\tfrac{d}{dx}(\sec(\sec^{-1} x)) &= \tfrac{d}{dx} x \\
(\sec(\sec^{-1} x)\tan(\sec^{-1} x))\tfrac{d}{dx}\sec^{-1} x &= 1 \\
\tfrac{d}{dx}\sec^{-1} x &= \frac{1}{\sec(\sec^{-1} x)\tan(\sec^{-1} x)} \\
\tfrac{d}{dx}\sec^{-1} x &= \begin{cases} \frac{1}{x\sqrt{x^2-1}}, & x > 1 \\ \frac{1}{x(-\sqrt{x^2-1})}, & x < -1 \end{cases} \\
\tfrac{d}{dx}\sec^{-1} x &= \frac{1}{|x|\sqrt{x^2-1}}
\end{aligned}$$

$$\boxed{\tfrac{d}{dx}\sec^{-1} x = \tfrac{1}{|x|\sqrt{x^2-1}} \text{ for } x \in (-\infty, -1) \cup (1, \infty)}$$

Derivatives of inverse functions (memorize)

f	f'	domain
$\ln x$	$\frac{1}{x}$	$(0, \infty)$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$(-1, 1)$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$	$(-1, 1)$
$\arctan x$	$\frac{1}{1+x^2}$	$\mathbf{R}$
$\cot^{-1} x$	$\frac{-1}{1+x^2}$	$\mathbf{R}$
$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$(-\infty, -1) \cup (1, \infty)$
$\csc^{-1} x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$(-\infty, -1) \cup (1, \infty)$

Example

Compute $\frac{d}{dx} \ln |x|$

$$\begin{aligned}
\tfrac{d}{dx} \ln |x| &= \tfrac{d}{dx} \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases} \\
&= \begin{cases} \tfrac{d}{dx} \ln x, & x > 0 \\ \tfrac{d}{dx} \ln(-x), & x < 0 \end{cases} \\
&= \begin{cases} \tfrac{1}{x}, & x > 0 \\ \tfrac{1}{-x} \cdot (-1), & x < 0 \end{cases} \\
&= \begin{cases} \tfrac{1}{x}, & x > 0 \\ \tfrac{1}{x}, & x < 0 \end{cases} \\
&= \frac{1}{x}
\end{aligned}$$

$\boxed{\frac{d}{dx} \ln |x| = \frac{1}{x} \text{ for } x \neq 0}$

Key rule for exponentials

$$a^b = e^{b \ln a} \quad \text{if } a > 0 \text{ and } b \in \mathbf{R}$$

Quick Justification

$x = e^{\ln x}$ so $a^b = e^{\ln a^b} = e^{b \ln a}$

General Power Rule

Supppose $r \in \mathbf{R}$ and $x > 0$. Then

$$\begin{aligned}
\tfrac{d}{dx} x^r &= \tfrac{d}{dx} e^{r \ln x} \\
&= e^{r \ln x} \cdot \frac{r}{x} \\
&= x^r \cdot \frac{r}{x} \\
&= r x^{r-1}
\end{aligned}$$

$\boxed{\frac{d}{dx} x^r = r x^{r-1} \text{ for } r \in \mathbf{R} \text{ a constant and } x > 0}$

Derivative of exponentials

Supppose $a > 0$. Then

$$\begin{aligned} \frac{d}{dx} a^x &= \frac{d}{dx} e^{x \ln a} \\ &= e^{x \ln a} \cdot \ln a \\ &= a^x \cdot \ln a \end{aligned}$$

$\boxed{\frac{d}{dx} a^x = (a^x \cdot \ln a) \text{ for any constant } a > 0.}$

Derivative of logs

Supppose $a > 0$ and $a \neq 1$. Then

$$\begin{aligned} \frac{d}{dx} \log_a x &= \frac{d}{dx} \frac{\ln x}{\ln a} \\ &= \frac{1}{x \ln a} \end{aligned}$$

$\boxed{\frac{d}{dx} \log_a x = \frac{1}{x \ln a} \text{ for any constant } a \in (0, 1) \cup (1, \infty)}$

Derivative e^x two ways

$\frac{d}{dx} e^x = x e^{x-1}$ WRONG!!! Power rule only works for constant exponents

$$\frac{d}{dx} e^x = e^x \quad \checkmark$$

Derivative x^x two ways

$\frac{d}{dx} x^x = x x^{x-1}$ WRONG!!! Power rule only works for constant exponents

$$\begin{aligned} \frac{d}{dx} x^x &= \frac{d}{dx} \left(e^{x \ln x} \right) \\ &= e^{x \ln x} \left(\ln x + x \cdot \frac{1}{x} \right) \\ &= x^x (\ln x + 1) \quad \checkmark \end{aligned}$$

Logarithmic Differentiation

Logarithmic Differentiation

If $f(x)$ is a complicated product, quotient or power

1. compute $\frac{d}{dx} \ln |f(x)| = \frac{f'(x)}{f(x)}$
2. solve for $f'(x) = f(x) \frac{d}{dx} \ln |f(x)|$

Example (Logarithmic differentiation)

Compute $\frac{\mathrm{d}}{\mathrm{d}x}\frac{(x+2)^8\sqrt[3]{\tan x}}{\sin x}$.

$$\begin{aligned} f(x) &= \frac{(x+2)^8\sqrt[3]{\tan x}}{\sin x} \\ \ln|f(x)| &= \ln\frac{|x+2|^8\sqrt[3]{|\tan x|}}{|\sin x|} \\ \ln|f(x)| &= 8\ln|x+2| + \tfrac{1}{3}\ln|\tan x| - \ln|\sin x| \end{aligned}$$

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \tfrac{\mathrm{d}}{\mathrm{d}x}\left(8\ln|x+2| + \tfrac{1}{3}\ln|\tan x| - \ln|\sin x|\right) \\ &= \frac{8}{x+2} + \frac{\sec^2 x}{3\tan x} - \frac{\cos x}{\sin x} \end{aligned}$$

$$f'(x) = \frac{(x+2)^8\sqrt[3]{\tan x}}{\sin x}\cdot\left(\frac{8}{x+2} + \frac{\sec^2 x}{3\tan x} - \frac{\cos x}{\sin x}\right)$$

Example

Compute $\frac{\mathrm{d}}{\mathrm{d}x}\left[(3x^3+x-1)^{9x}(\sec x)^4\sqrt[5]{x+\cot^2 x}+2x\right]$.

$$\begin{aligned} f(x) &= (3x^3+x-1)^{9x}(\sec x)^4\sqrt[5]{x+\cot^2 x} \\ \ln|f(x)| &= \ln\left|(3x^3+x-1)^{9x}(\sec x)^4\sqrt[5]{x+\cot^2 x}\right| \\ \ln|f(x)| &= (9x)\ln|3x^3+x-1| + 4\ln|\sec x| + \tfrac{1}{5}\ln|x+\cot^2 x| \end{aligned}$$

$$\frac{f'(x)}{f(x)} = 9\ln|3x^3+x-1| + \frac{9x(9x^2+1)}{3x^3+x-1} + \frac{4\sec x\tan x}{\sec x} + \frac{1+2\cot x(-\csc^2 x)}{5(x+\cot^2 x)}$$

Example (*continued*)

$$f'(x) = \left((3x^3 + x - 1)^{9x}(\sec x)^4 \sqrt[5]{x + \cot^2 x}\right)$$
$$\cdot \left(9 \ln|3x^3 + x - 1| + \frac{9x(9x^2+1)}{3x^3 + x - 1} + 4\tan x + \frac{1 - 2\cot x \csc^2 x}{5(x + \cot^2 x)}\right)$$

$$\tfrac{\mathrm{d}}{\mathrm{d}x}[(3x^3 + x - 1)^{9x}(\sec x)^4 \sqrt[5]{x + \cot^2 x} + 2x]$$
$$= \left((3x^3 + x - 1)^{9x}(\sec x)^4 \sqrt[5]{x + \cot^2 x}\right)$$
$$\cdot \left(9 \ln|3x^3 + x - 1| + \frac{9x(9x^2+1)}{3x^3 + x - 1} + 4\tan x + \frac{1 - 2\cot x \csc^2 x}{5(x + \cot^2 x)}\right)$$
$$+ 2$$