Lecture 14 – September 26, 2018

- WH 3 now posted – Due Tues. Oct. 2, 2018
- Quiz 4 tomorrow

Today

- Differentiation summary
- Related rates

Differentiation Summary

Basic derivatives (memorize)

$\frac{d}{dx}c = 0$	$\frac{d}{dx}x^r = rx^{r-1}$ if $x > 0$	$\frac{d}{dx}e^x = e^x$
$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\sec x = \sec x \tan x$
$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\cot x = -\csc^2 x$	$\frac{d}{dx}\csc x = -\csc x \cot x$
$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$	$\frac{d}{dx}\sec^{-1} x = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$	$\frac{d}{dx}\csc^{-1} x = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
$\frac{d}{dx}\ln x = \frac{1}{x}$		

Derivative rules

If f and g are differentiable at x then:

Sum rule	$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$
Product rule	$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
Quotient rule	$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$ assuming $g(x) \neq 0$
Chain rule	$\frac{d}{dx}f(g(x)) = f'(g(x))f(x)$ assuming f is diff. at $g(x)$
Inverse function rule	$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$ assuming f is diff. at $f^{-1}(x)$ and $f'(f^{-1}(x)) \neq 0$

Miscellaneous techniques

Implicit differentiation	1. take $\frac{d}{dx}$ of both sides of equation 2. put terms with y' on one side 3. solve for y'
Definition of a^b	$a^b = e^{\ln(a^b)} = e^{b \ln a}$ if $a > 0$
Definition of $\log_a b$	$\log_a b = \frac{\ln b}{\ln a}$ if $a > 0$ and $b > 0$
Logarithmic differentiation	1. start with equation $y = f(x)$ 2. take ln of both sides of equation 3. take $\frac{d}{dx}$ of both sides of equation 4. solve for y'

Related Rates Problems

Related Rates

If y is a quantity of interest related to a quantity x, how does the rate of change of y depend on x and the rate of change of x?

- Give equation relating x and y.
- Take derivative with respect to proper variable (generally t for time)
- Rate of change of x is $\frac{dx}{dt}$
- Rate of change of y is $\frac{dy}{dt}$
- Tough part of these problems is getting correct equation relating y and x

Example

A trough has a triangular cross section which is $2\mathrm{m}$ high, $2\mathrm{m}$ wide. The trough is $4\mathrm{m}$ long. Water is poured in at a rate of $\frac{1}{2}\mathrm{m}^3/\mathrm{min}$. How fast is the water level rising when the depth is $1\mathrm{m}$?

Solution:

- Draw picture with changing quantities labeled with **variables**.
- Give equation relating relevant quantities. $V = \frac{1}{2}bh\ell$
- $\ell = 4$ is not changing.
- using similar triangles $\frac{h}{b} = \frac{2}{2}$ so $2h = 2b$ so $b = h$
- $V = \frac{1}{2}hh4$
- $V = 2h^2$
- $\frac{dV}{dt} = 4h\frac{dh}{dt}$ so $\frac{dh}{dt} = \frac{\left(\frac{dV}{dt}\right)}{4h}$
- Now plug in $h = 1\mathrm{m}$ and $\frac{dV}{dt} = \frac{1}{2}\mathrm{m}^3/\mathrm{min}$
- $\frac{dh}{dt} = \frac{\left(\frac{dV}{dt}\right)}{4h} = \frac{\left(\frac{1}{2}\right)}{4\cdot 1} = \boxed{\frac{1}{8}\,\mathrm{m}/\mathrm{min}}$

Helpful geometry

Helpful geometry

Circumference of circle	$2\pi r$	Area of circle	πr^2
Length of sector	$r\theta$	Area of sector	$\frac{1}{2}r^2\theta$
Area of rectangle	$\ell \cdot w$	Area of triangle	$\frac{1}{2}bh$
Area of trapezoid	$\frac{1}{2}(a+b)h$	Area of annulus	$\pi(R^2 - r^2)$
Volume of cylinder	$\pi r^2 h$	Volume of prism	Ah
Volume of cone	$\frac{1}{3}\pi r^2 h$	Volume of cone	$\frac{1}{3}Ah$
Surface area of sphere	$4\pi r^2$	Volume of sphere	$\frac{4}{3}\pi r^3$

Helpful Trigonometry

Helpful Trigonometry

- Similar triangles
- Law of sines $\frac{\sin\alpha}{A} = \frac{\sin\beta}{B} = \frac{\sin\gamma}{C}$
- Law of cosines $A^2 = B^2 + C^2 - 2BC\cos\alpha$

Example

An inverted conical tank with height $3\mathrm{m}$ and radius $1\mathrm{m}$ is filled at a rate of $\frac{1}{3}\mathrm{m}^3/\mathrm{min}$. How fast is the water level rising when $\frac{1}{2}\mathrm{m}^3$ has been poured into the tank?
Solution:

- Draw picture with changing quantities labeled with **variables**.
- $V = \frac{1}{3}\pi r^2 h$
- using similar triangles $\frac{r}{h} = \frac{1}{3}$ so $r = \frac{h}{3}$
- $V = \frac{1}{3}\pi(\frac{h}{3})^2 h = \frac{\pi}{27}h^3$
- $h = \sqrt[3]{\frac{27}{\pi}V} = \frac{3}{\sqrt[3]{\pi}}V^{\frac{1}{3}}$
- $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{3}{\sqrt[3]{\pi}}\frac{1}{3}V^{-2/3} \cdot \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{\sqrt[3]{\pi}}V^{-2/3} \cdot \frac{\mathrm{d}V}{\mathrm{d}t}$
- Now plug in $V = \frac{1}{2}\,\mathrm{m}^3$ and $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{3}\,\mathrm{m}^3/\mathrm{min}$
- $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{\sqrt[3]{\pi}}\left(\frac{1}{2}\right)^{-2/3} \cdot \left(\frac{1}{3}\right) = \boxed{\sqrt[3]{\frac{4}{27\pi}}\,\mathrm{m}/\mathrm{min}}$

Example

A rotating wheel of radius 1m is connected to a 3m crankshaft and then to a piston. The wheel is rotating **clockwise** at a rate of $5\,\text{rpm}$. What is the **speed** of the piston when the angle made by the the axis of the piston, the center of the flywheel and the connection of the flywheel to the crankshaft is $\frac{\pi}{4}$?

- Draw picture with changing quantities labeled with **variables**.
- $x = \cos\theta + \sqrt{3^2 - \sin^2\theta} = \cos\theta + (9 - \sin^2\theta)^{1/2}$
- $\frac{dx}{dt} = -\sin\theta \cdot \frac{d\theta}{dt} + \frac{1}{2}(9 - \sin^2\theta)^{-1/2}(0 - 2\sin\theta\cos\theta \cdot \frac{d\theta}{dt})$
- What is $\frac{d\theta}{dt}$?
  $5\ \text{rpm (clockwise)} = -2\pi \cdot 5\ \text{rad/min} = -10\pi\ \text{rad/min}$
- $\frac{dx}{dt} = -\sin\frac{\pi}{4} \cdot (-10\pi) + \frac{1}{2}(9 - \sin^2\frac{\pi}{4})^{-1/2}(-2\sin\frac{\pi}{4}\cos\frac{\pi}{4}) \cdot (-10\pi)$
- $\frac{dx}{dt} = \frac{\sqrt{2}}{2} \cdot 10\pi + \frac{1}{2}(9 - (\frac{\sqrt{2}}{2})^2)^{-1/2}(2(\frac{\sqrt{2}}{2})^2) \cdot (10\pi)$
- $\frac{dx}{dt} = 5\sqrt{2}\pi + \left(\frac{1}{2}\right)\left(\frac{17}{2}\right)^{-1/2} \cdot 10\pi = 5\sqrt{2}\pi + \sqrt{\frac{2}{17}} \cdot 5\pi$
- speed $= |\frac{dx}{dt}| = \boxed{5\pi\sqrt{2} + \frac{5\pi\sqrt{2}}{\sqrt{17}}\ \text{m/min}} \approx 27.6\,\text{m/min}$

Example

An actor 2m tall walks at $\frac{1}{3}\text{m/s}$ away from a wall and toward a floor light 4m from the wall. How fast does his shadow on the wall behind him grow when he is 3m from the wall?
Solution:

- Draw picture with changing quantities labeled with **variables**.
- using similar triangles $\frac{2}{4-x} = \frac{h}{4}$ so $h = 8(4-x)^{-1}$
- $\frac{dh}{dt} = -8(4-x)^{-2} \cdot (-1) \cdot \frac{dx}{dt} = 8(4-x)^{-2}\frac{dx}{dt}$
- Now plug in $x = 3\text{m}$ and $\frac{dx}{dt} = \frac{1}{3}\text{m/s}$
- $\frac{dh}{dt} = 8(4-x)^{-2}\frac{dx}{dt} = 8(4-3)^{-2}\frac{1}{3} = \boxed{\frac{8}{3}\text{m/s}}$

Example

The hour hand on a clock has length $1\,\mathrm{m}$ and minute hand has length $2\,\mathrm{m}$. How fast is distance between the tips of the hands changing at 3pm?

- $\theta =$ angle between hour hand and minute hand
- $x =$ distance between tip of hour hand and tip of minute hand
- By Law of Cosines

$$x^2 = 1^2 + 2^2 - 2 \cdot 1 \cdot 2 \cos\theta = 5 - 4\cos\theta$$

- $2x\frac{dx}{dt} = 4\sin\theta\frac{d\theta}{dt}$
- $\theta = \theta_m - \theta_h$
- $\frac{d\theta}{dt} = \frac{d\theta_m}{dt} - \frac{d\theta_h}{dt} = -2\pi - \left(-\frac{2\pi}{12}\right) = -\frac{11\pi}{6}\,\mathrm{rad/h}$
- At 3pm $\theta = \frac{\pi}{2}$.
- At 3pm $x = \sqrt{1^2 + 2^2} = \sqrt{5}$
- So $\frac{dx}{dt} = \frac{4\sin\theta\frac{d\theta}{dt}}{2x} = \frac{4(\sin\frac{\pi}{2})(-\frac{11\pi}{6})}{2\cdot\sqrt{5}} = \boxed{-\frac{11\pi}{3\sqrt{5}}\,\mathrm{m/h}}$