Lecture 15 – September 28, 2018

- WH 3 now posted – Due Tues. Oct. 2, 2018

Today

- Absolute Extrema
- Extreme Value Theorem
- Local Extrema
- Critical Points
- Intervals of Increase/Decrease

Absolute Extrema

Definition (Absolute maxima, minima and extrema)

A function f with domain D has an **absolute maximum** of $f(c)$ at c if for all $x \in D$ we have $f(x) \leq f(c)$. The function f has an **absolute minimum** of $f(c)$ at c if for all $x \in D$ we have $f(x) \geq f(c)$. The function f has an **absolute extremum** at c it has an absolute maximum or absolute minimum at c.

Example

Not all functions have an absolute maximum and minimum

- $f(x) = x^2$ has an absolute minimum of 0 at $x = 0$ and no absolute maximum on its domain ($\mathbf{R}$).
- $g(x) = -x^2$ has an absolute maximum of 0 at $x = 0$ and no absolute minimum on its domain ($\mathbf{R}$).
- $h(x) = x(x-1)(x-2)$ has no absolute maximum or minimum on its domain ($\mathbf{R}$).
- $j(x) = \sin x$ has an absolute maximum of 1 at $x = \frac{\pi}{2} + 2\pi n$ for all $n \in \mathbf{Z}$ and an absolute minimum of -1 at $x = \frac{3\pi}{2} + 2\pi n$ for all $n \in \mathbf{Z}$
- $k(x) = e^x$ has no absolute maximum or minimum on its domain ($\mathbf{R}$).

Example

Not all functions have absolute extrema

- $\ell(x) = 2x$ with domain $[-1, 1)$ has an absolute minimum of -2 at $x = -1$ and no absolute maximum.
- $w(x) = 2x$ with domain $[-1, 1]$ has an absolute minimum of -2 at $x = -1$ and an absolute maximum of 2 at $x = 1$.
- $v(x) = \begin{cases} x, & 0 \le x \le 1 \\ 5 - x, & 1 < x \le 2 \end{cases}$ on the implicit domain $[0, 2]$ has an absolute minimum of 0 at $x = 0$ and no absolute maximum.
- $z(x) = \begin{cases} \sin 3x, & 0 \le x \le 2\pi \\ 6\cos x, & 2\pi < x \le 4\pi \end{cases}$ on the implicit domain $[0, 4\pi]$ has an absolute minimum of -6 at $x = 3\pi$ and an absolute maximum of 6 at $x = 4\pi$.

Theorem (Extreme Value Theorem (EVT))

If f is **continuous** *on the closed interval $[a, b]$ then f has an absolute maximum and absolute minimum on the interval $[a, b]$*

Note: This is a prototypical existence theorem providing no clue as to how to find the absolute max or min which is purported to exist.

Local Extrema

Definition (Local maxima and minima)

A function f has a **local maximum** of $f(c)$ at c if there is an open interval I containing c such that for all $x \in I$ we have $f(x) \le f(c)$. The function f has a **local minimum** of $f(c)$ at c if there is an open interval I containing c such that for all $x \in I$ we have $f(x) \ge f(c)$. The function f has a **local extremum** at c it has a local maximum or local minimum at c.

Example

Mark the locations of the local extrema for the functions $f, g, h, j, k, \ell, w, v, z$ above

Finding local extrema

Definition (Critical point)

If

1. c is an **interior point** in the domain of f
2. and $f'(c) = 0$ **or** if f is not differentiable at c

then c is a **critical point** of f.

Theorem (Critical Point Theorem)

If the function f has a local extremum at c then c is a critical point of f.

Example

- $f(x) = |x| + 1$ has a local minimum of 1 at 0 and f is not differentiable at 0
- $g(x) = -x^2 + 1$ has a local maximum of 1 at 0 and $f'(0) = 0$.
- $h(x) = \sin x$ has a local maximum of 1 at $\frac{\pi}{2}$ and $g'(\frac{\pi}{2}) = 0$

Local maxima and minima

The converse is not generally true. Just because c is a critical point of f we may **not** conclude that the function f has a local extremum at c.

Example

- 0 is a critical point of $f(x) = \sqrt[3]{x}$ but f does not have a local extremum at 0.
- 0 is a critical point of $g(x) = x^3$ but g does not have a local extremum at 0.
- Every real number $c \in \mathbf{R}$ is a critical point of $h(x) = 10$ and the local maximum 10 and local minimum of 10 occurs at every $c \in \mathbf{R}$.

Finding absolute maxima and minima of continuous functions on closed intervals

1. Find critical points
2. Evaluate value of function at **critical points** and **endpoints**
3. Choose max value

Example

Find the absolute max and min of the function $f(x) = \frac{x^4}{4} - x^3 + x^2 - 6$ on the interval $[-1, 3]$.
Solution:

- By EVT and Critical Point Theorem absolute maximum and minimum of f on interval $[-1, 3]$ exist and must occur at critical points or endpoints.
- $f'(x) = x^3 - 3x^2 + 2x = x(x-1)(x-2)$
- Critical points: 0, 1 and 2

$$f(0) = \frac{0^4}{4} - 0^3 + 0^2 - 6 = -6 = \boxed{-\frac{24}{4} \text{ abs. min.}}$$
$$f(1) = \frac{1^4}{4} - 1^3 + 1^2 - 6 = -\frac{23}{4}$$
$$f(2) = \frac{2^4}{4} - 2^3 + 2^2 - 6 = -6 = \boxed{-\frac{24}{4} \text{ abs. min.}}$$

- Endpoints: -1 and 3

$$f(-1) = \frac{(-1)^4}{4} - (-1)^3 + (-1)^2 - 6 = \boxed{-\frac{15}{4} \text{ abs. max.}}$$
$$f(3) = \frac{3^4}{4} - 3^3 + 3^2 - 6 = \boxed{-\frac{15}{4} \text{ abs. max.}}$$

Intervals of increase/decrease

Definition (Intervals of increase/decrease)

1. A function f is **increasing** on an interval I if for all $x_1, x_2 \in I$ if $x_1 < x_2$ then $f(x_1) < f(x_2)$.
2. A function f is **decreasing** on an interval I if for all $x_1, x_2 \in I$ if $x_1 < x_2$ then $f(x_1) > f(x_2)$.
3. A function f is **nondecreasing** on an interval I if for all $x_1, x_2 \in I$ if $x_1 < x_2$ then $f(x_1) \leq f(x_2)$.
4. A function f is **nonincreasing** on an interval I if for all $x_1, x_2 \in I$ if $x_1 < x_2$ then $f(x_1) \geq f(x_2)$.

Example

- $f(x) = x^2$ is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$.
- $k(x) = e^x$ is increasing on $\mathbf{R}$.
- $b(x) = 2$ is nonincreasing (and nondecreasing) on $\mathbf{R}$.

- Definitions for increasing and decreasing are difficult to check (infinitely many pairs of points x_1, x_2 to compare)
- Following two theorems say that for differentiable functions, finding intervals of increase/decrease amounts to checking sign of derivative

Theorem (Positive derivative implies increasing)

If f is **continuous** *on the interval I and $f'(x) > 0$ for all interior points x of I then f is increasing on I.*

Theorem (Negative derivative implies decreasing)

If f is **continuous** *on the interval I and $f'(x) < 0$ for all interior points x of I then f is decreasing on I.*

Increasing is not same as $f'(x) > 0$

A function may be increasing even though $f'(x) = 0$ or f is not differentiable

- $f(x) = x^3$ is increasing on **R** even though $f'(0) = 0$
- $f(x) = \sqrt[3]{x}$ is increasing on **R** even though f not differentiable at 0