Lecture 16 – October 1, 2018

- WH 3 – Due tomorrow

Today

- First derivative test for increase/decrease
- First derivative test for local extrema
- Concavity and Inflection points
- Second derivative test for concavity
- Second derivative test for local extrema
- Symmetry of functions (even, odd, periodic)

First derivative test for increase/decrease

- Definitions for increasing and decreasing are difficult to check (infinitely many pairs of points x_1, x_2 to compare)
- Following two theorems say that for differentiable functions, finding intervals of increase/decrease amounts to checking sign of derivative

Theorem (Positive derivative implies increasing)

If f is **continuous** *on the interval I and $f'(x) > 0$ for all interior points x of I then f is increasing on I.*

Theorem (Negative derivative implies decreasing)

If f is **continuous** *on the interval I and $f'(x) < 0$ for all interior points x of I then f is decreasing on I.*

Increasing is not same as $f'(x) > 0$

A function may be increasing even though $f'(x) = 0$ or f is not differentiable

- $f(x) = x^3$ is increasing on $\mathbf{R}$ even though $f'(0) = 0$
- $f(x) = \sqrt[3]{x}$ is increasing on $\mathbf{R}$ even though f not differentiable at 0

Example

Find the intervals of increase and decrease for $f(x) = \frac{x^2}{x^2-1}$

$$f'(x) = \frac{2x(x^2-1) - x^2(2x)}{(x^2-1)^2} = \frac{2x^3 - 2x - 2x^3}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2}$$

Thus $f'(x) > 0$ if $x \in (-\infty, -1)$ or $x \in (-1, 0)$
Largest intervals of increase: $(-\infty, -1)$ and $(-1, 0]$
Thus $f'(x) < 0$ if $x \in (-1, 0)$ or $x \in (1, \infty)$
Largest intervals of decrease: $[0, 1)$ and $(1, \infty)$

First derivative test for local extrema

Theorem (First derivative test for local extrema)

If

1. *c is critical point of f,*
2. *f is continuous on an open interval containing c*
3. *and f is differentiable on an open interval containing c except possibly at c*

then

- *if $f'(x)$ changes from* **positive to negative** *at c then f has a* **local maximum** *at c.*
- *if $f'(x)$ changes from* **negative to positive** *at c then f has a* **local minimum** *at c.*
- *if $f'(x)$ is positive* **before and after** *c then f does not have a local extremum at c.*
- *if $f'(x)$ is negative* **before and after** *c then f does not have a local extremum at c.*

Example

Find all local extrema of the function $h(x) = x^2$ and decide if they are local maxima or minima

- $h'(x) = 2x$.
- Only critical point of h is $x = 0$.
- For $x < 0$ we have $h'(x) = 2x < 0$ and for $x > 0$ we have $h'(x) = 2x > 0$.
- Thus the first derivative test for local extrema tells us that h has a local minimum of $h(0) = 0$ at $x = 0$.

Theorem (Single local extremum implies absolute extremum)

If f is continuous on an interval I with only one local extremum ocuring only at c then

- *if f has a local maximum at c then f has an absolute maximum (for the interval I) at c.*
- *if f has a local minimum at c then f has an absolute minimum (for the interval I) at c.*

Example

Previous example showed $h(x) = x^2$ has single local minimum occuring only at 0

- By theorem above h has an absolute minimum on the interval $(-\infty, \infty)$ at $x = 0$.

Concavity

Definition (Concavity)

- If f is differentiable on the interval I and f' is increasing on I then f is **concave up** on I.
- If f is differentiable on the interval I and f' is decreasing on I then f is **concave down** on I.
- If f changes concavity (up to down or down to up) at c then f has an **inflection point at** c.

Example

Let $q(x) = x^4$. Find the largest intervals on which q is concave up.

- $q'(x) = 4x^3$.
- $4x^3$ is increasing on $(-\infty, 0]$ by first derivative test for increase.
- $4x^3$ is increasing on $[0, \infty)$ by first derivative test for increase.
- $4x^3$ is continuous at 0.
- Thus $q'(x) = 4x^3$ is increasing on $(-\infty, \infty)$
- Thus $q(x) = x^4$ is concave up on $(-\infty, \infty)$

Theorem (Second derivative test for concavity)

If f'' exists on the open interval I then

- *If $f''(x) > 0$ on I then f is* **concave up** *on I.*
- *If $f''(x) < 0$ on I then f is* **concave down** *on I.*

Example

1. $f(x) = e^x$. Then $f''(x) = e^x > 0$ for all $x \in \mathbf{R}$ so f is concave up on $\mathbf{R}$.
2. $g(x) = \ln x$. Then $g''(x) = -\frac{1}{x^2} < 0$ for all $x > 0$ so f is concave down on $(0, \infty)$.
3. If $q(x) = x^4$ then $q''(x) = 12x^2$. By second derivative test for concavity we get that q is concave up on $(-\infty, 0)$ and $(0, \infty)$. But we saw in previous example that definition of concave up shows that q is concave up on $(-\infty, \infty)$.

Second derivative test for local extrema

Theorem (Second derivative test for local extrema)

If f'' exists on an open interval containing the critical point c then

- *If $f''(c) > 0$ then f has a local minimum at c.*
- *If $f''(c) < 0$ then f has a local maximum at c.*
- *If $f''(c) = 0$ then the test is inconclusive.*

Example

1. $h(x) = x^2$. Then $h''(x) = 2 > 0$ for all $x \in \mathbf{R}$ so h is concave up on $\mathbf{R}$. Also, $x = 0$ is unique critical point. By second derivative test h has local min at 0 and by "Single local extremum theorem" $h(0) = 0$ is absolute min. for h on $\mathbf{R}$.

Symmetries of functions

Definition (Odd and even functions)

- If $f(-x) = -f(x)$ for all x in the domain of f then f is an **odd function**.
- If $f(-x) = f(x)$ for all x in the domain of f then f is an **even function**.

Example

Decide if the following functions are odd, even, both or neither.

- $f(x) = 3x^2 + 11$

$$f(-x) = 3(-x)^2 + 11 = 3x^2 + 11 = f(x) \quad \text{so } f \text{ is even}$$

- $g(x) = \sin(2x) + x\cos 7x$

$$g(-x) = \sin(2(-x)) + (-x)\cos 7(-x) = -\sin(2x) - x\cos 7x = -g(x)$$

  so g is odd

Example

Decide if the following functions are odd, even, both or neither.

- $h(x) = 3x + x^2$

$$h(1) = 3(1) + 1^2 = 4$$
$$h(-1) = 3(-1) + (-1)^2 = -2$$

  so h is not even or odd

- $j(x) = 0$

$$j(-x) = 0 = j(x)$$
$$j(-x) = 0 = -0 = -j(x)$$

  so j is even and odd.

Definition (Periodic function)

The function f is **periodic with period** $k > 0$ if for all x in the domain of f we have

$$f(x + k) = f(x)$$

Example

Notice that $\sin x$ is periodic with period 2π since

$$\sin(x + 2\pi) = \sin x$$

However, $\sin x$ is also periodic with period 20π since

$$\sin(x + 20\pi) = \sin x$$

Definition (Periodic function)

The function f is periodic with **fundamental period** $k > 0$ if k is the smallest positive number such that

$$f(x + k) = f(x)$$

for all x in the domain of f

Example

1. $f(x) = \sin x$ is periodic with fundamental period 2π
2. $g(x) = 4$ is periodic with period k for any $k > 0$ since

$$g(x + k) = 4 = g(x).$$

   Thus g is periodic but does not have a fundamental period.