Lecture 18 – October 5, 2018

- Midterm 2 – Thurs. 10/18 6pm-6:55pm in Journalism Building (JR) 300

Today

- Minimization/Maximization

Minimization/Maximization

Minimization/Maximization Problems

These problems ask you to minimize or maximize a quantity given certain constraints.

1. Draw a picture (if applicable)
2. Give a function for the quantity you want to maximize
3. Eliminate all but one variable using geometry and/or other problem info
4. Give the **appropriate domain** for your function (may differ from implicit domain)
5. Does **EVT** apply? (Is domain a closed interval and the function is continuous on the domain?)
6. Find critical points.
7. Does **single local extremum** theorem apply?
8. compute value of function on critical points **in domain** and at **endpoints**
9. Choose largest (smallest) value of f **in domain**. Don't forget **units**

Example (Maximizing area inside triangular fence)

An isosceles triangular region is enclosed by a fence on the two equal sides and a long wall on the third side. What is the maximum area that can be enclosed with $20\mathrm{m}$ of fence?

1. PICTURE
2. $A = \frac{1}{2}bh$

$$\begin{aligned} h^2 + \left(\frac{b}{2}\right)^2 &= \left(\frac{20}{2}\right)^2 \\ \left(\frac{b}{2}\right)^2 &= 100 - h^2 \\ \frac{b}{2} &= \sqrt{100 - h^2} \\ b &= 2\sqrt{100 - h^2} \end{aligned}$$

Example (*continued*)

3.

$$A = \tfrac{1}{2}bh$$
$$A(h) = \tfrac{1}{2} \cdot 2\sqrt{100 - h^2}(h)$$
$$A(h) = h(100 - h^2)^{\frac{1}{2}}$$

4. Domain: $h \in [0, 10]$ NOTE: Implicit domain would have been $[-10, 10]$
5. A is cont. on $[0, 10]$ so by EVT has absolute max and min.

Example (*continued*)

6. Critical points

$$\begin{aligned}\tfrac{dA}{dh} &= (100 - h^2)^{\frac{1}{2}} + h \cdot \tfrac{1}{2}(100 - h^2)^{-\frac{1}{2}}(-2h) \\ &= \frac{100 - h^2}{\sqrt{100 - h^2}} - \frac{h^2}{\sqrt{100 - h^2}} \\ &= \frac{100 - 2h^2}{\sqrt{100 - h^2}}\end{aligned}$$

Critical points from numerator $100 - 2h^2 = 0$ so $h^2 = 50$ so $h = \pm\sqrt{50}$ NOTE $7^2 < 50 < 8^2$ so $7 < \sqrt{50} < 8$
Critical points from denominator: $\sqrt{100 - h^2} = 0$ so $100 - h^2 = 0$ so $h = \pm\sqrt{100} = \pm 10$

7. Single local extremum at $\sqrt{50}$ on domain $[0, 10]$ applies.

Example (*continued*)

8. Critical points in domain $h = \sqrt{50}$, $h = 10$, endpoints $h = 0$, $h = 10$

$$A(0) = 0\sqrt{100 - h^2} = 0$$

$$A(\sqrt{50}) = \sqrt{50} \cdot \sqrt{100 - \sqrt{50}^2} = \sqrt{50} \cdot \sqrt{50} = 50$$

$$A(10) = 10 \cdot \sqrt{100 - 10^2} = 10 \cdot 0 = 0$$

9. Maximum area enclosable with $20\mathrm{m}$ of fence is $50\mathrm{m}^2$

Example (Maximizing revenue)

Demand for a certain computer at price $p\,\mathrm{dollars}$ is

$$D(p) = 1000(1000 - p)\,\mathrm{computers}$$

Find the price (in $\mathrm{dollars}$) at which revenue from this computer will be maximized. What is this maximum revenue?

2. Revenue at price p is $R(p) = pD = 1000(1000 - p)p$
3. Eliminating demand D we get that revenue is

$$R(p) = 1000(1000 - p)p$$

4. Domain: $p \in [0, 1000]$ (NOTE: Implicit domain would have been $\mathbf{R}$)
5. R is cont. on $[0, 1000]$ so EVT applies.

Example (*continued*)

6. Critical points

$$\begin{aligned}\frac{\mathrm{d}R}{\mathrm{d}p} &= 1000(-1)p + 1000(1000 - p) \\ &= 1000(1000 - 2p)\end{aligned}$$

   Critical point when $1000 - 2p = 0$ so $p = 500$

7. Single local extremum at $p = 500$ so single local extremum theorem applies.

8.

$$\begin{aligned}R(0) &= 1000(1000 - 0)0 = 0 \\ R(500) &= 1000(1000 - 500)500 = 250,000,000 \\ R(1000) &= 1000(1000 - 1000)1000 = 0\end{aligned}$$

9. Maximum revenue of $R = \$250,000,000$ when price is $p = \$500$.

Example (Maximizing viewing angle)

The bottom of the projection screen in a theater is $1\mathrm{m}$ above the head of a viewer. The screen itself is $2\mathrm{m}$ tall. Assuming that the seats all have the same elevation (no stadium seating) how far should the viewer sit from the screen in order to maximize her viewing angle?

1. PICTURE
2.

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{3}{x} \\ \alpha + \beta &= \arctan\frac{3}{x} \\ \alpha &= \arctan\frac{3}{x} - \beta\end{aligned}$$

3. $\tan(\beta) = \frac{1}{x}$ so $\beta = \arctan\frac{1}{x}$

$$\alpha(x) = \arctan\frac{3}{x} - \arctan\frac{1}{x}$$

Example (*continued*)

4. Domain: $[0, \infty)$
5. Can't use EVT since interval is not closed and bounded. BUT function is continuous except possibly at 0.
6. Critical points

$$
\begin{aligned}
\frac{d\alpha}{dx} &= \frac{1}{1+\left(\frac{3}{x}\right)^2} \cdot \left(-\frac{3}{x^2}\right) - \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right) \\
&= \frac{1}{1+\frac{9}{x^2}} \cdot \left(-\frac{3}{x^2}\right) - \frac{1}{1+\frac{1}{x^2}} \cdot \left(-\frac{1}{x^2}\right) \\
&= \frac{1}{\frac{x^2+9}{x^2}} \cdot \left(-\frac{3}{x^2}\right) - \frac{1}{\frac{x^2+1}{x^2}} \cdot \left(-\frac{1}{x^2}\right) \\
&= \frac{x^2}{x^2+9} \cdot \left(-\frac{3}{x^2}\right) - \frac{x^2}{x^2+1} \cdot \left(-\frac{1}{x^2}\right) \\
&= \frac{-3}{x^2+9} + \frac{1}{x^2+1}
\end{aligned}
$$

Example (*continued*)

6.

$$
\begin{aligned}
&= \frac{-3(x^2+1) + 1(x^2+9)}{(x^2+1)(x^2+9)} \\
&= \frac{-3x^2 - 3 + x^2 + 9}{(x^2+1)(x^2+9)} \\
&= \frac{-2x^2+6}{(x^2+1)(x^2+9)}
\end{aligned}
$$

Critical points from numerator $-2x^2 + 6 = 0$ so $x^2 = 3$ so $x = \pm\sqrt{3}$
Critical points from denominator: $(x^2+1)(x^2+9) = 0$ NO ROOTS

7. Single local extremum theorem applies

Example (*continued*)

8. Critical points in domain $x = \sqrt{3}$.
   - Can't use EVT so let's look at intervals of increase/ decrease
   - decreasing $(-\infty, -\sqrt{3}]$ and $[\sqrt{3}, \infty)$
   - increasing $[-\sqrt{3}, \sqrt{3}]$
   - On our domain must have max at $\alpha(\sqrt{3})$

$$\begin{aligned}\alpha(\sqrt{3}) &= \arctan \tfrac{3}{\sqrt{3}} - \arctan \tfrac{1}{\sqrt{3}} \\ &= \tfrac{\pi}{3} - \tfrac{\pi}{6} \\ &= \tfrac{\pi}{6}\end{aligned}$$

9. Maximum viewing angle when sitting $\sqrt{3}\,\mathrm{m}$ from screen wall. (about $1.73\,\mathrm{m}$)

Example (Where to swim across a canal)

A swimmer wants to walk along a canal 20m wide and and then swim across at an angle to end up at a spot across the canal and $40\mathrm{m}$ down the canal. He can swim at $1\mathrm{m/s}$ and run at $5\mathrm{m/s}$. At what point should he start swimming in order to minimize travel time? What is the minimum travel time?

1. PICTURE
2. Distance on land: $40 - x$
   Distance in water: $\sqrt{20^2 + x^2} = \sqrt{400 + x^2}$
   $d = rt$ so $t = \frac{d}{r}$
   Time on land: $t = \frac{d_\ell}{r_\ell} = \frac{40-x}{5}$
   Time in water: $t = \frac{d_w}{r_w} = \frac{\sqrt{400+x^2}}{1}$
3. Total time $T = \frac{40-x}{5} + \frac{\sqrt{400+x^2}}{1} = 8 - \frac{x}{5} + (400 + x^2)^{\frac{1}{2}}$
4. Domain: $x \in [0, 40]$
5. T is cont. on $[0, 40]$ so by EVT has absolute max and min.

Example (*continued*)

6. Critical points

$$
\begin{aligned}
\tfrac{\mathrm{d}T}{\mathrm{d}x} &= -\tfrac{1}{5} + \frac{1}{2}(400+x^2)^{-\frac{1}{2}} \cdot 2x \\
&= -\tfrac{1}{5} + \frac{x}{\sqrt{400+x^2}} \\
&= -\frac{\sqrt{400+x^2}}{5\sqrt{400+x^2}} + \frac{5x}{5\sqrt{400+x^2}} \\
&= \frac{5x - \sqrt{400+x^2}}{5\sqrt{400+x^2}}
\end{aligned}
$$

Critical points from numerator $5x - \sqrt{400+x^2} = 0$ so
$5x = \sqrt{400+x^2}$ so $25x^2 = 400 + x^2$ so $24x^2 = 400$ so $x = \pm\sqrt{\frac{50}{3}}$.
Only $x = \sqrt{\frac{50}{3}}$ is in domain.
Critical points from denominator: $400 + x^2 = 0$ so NO ROOTS

Example (*continued*)

7. Single local extremum theorem also applies. (Must show that T has local min at $x = \sqrt{\frac{50}{3}}$.)
8. Critical points in domain $x = \sqrt{\frac{50}{3}}$, endpoints $x = 0$, $x = 40$

$$
\begin{aligned}
T(0) &= \tfrac{40-0}{5} + \sqrt{400+0^2} = 8 + 20 = 28 \\
T\left(\sqrt{\tfrac{50}{3}}\right) &= \frac{40 - \sqrt{\frac{50}{3}}}{5} + \sqrt{400 + \sqrt{\tfrac{50}{3}}^{\,2}} \\
&= 8 + 24\sqrt{\tfrac{2}{3}} \approx 27.60 \\
T(40) &= \tfrac{40-40}{5} + \sqrt{400+40^2} = \sqrt{2000} \approx 44.72
\end{aligned}
$$

9. Start swimming $\sqrt{\frac{50}{3}}\mathrm{m} \approx 4.08\mathrm{m}$ upstream from destination to get a Minimum travel time of $8 + 24\sqrt{\frac{2}{3}}\mathrm{s}$