Lecture 19 – October 8, 2018

- Midterm 2 – Thurs. 10/18 6pm-6:55pm in Journalism Building (JR) 300

Today

- Linear Approximation

Linear Approximation

1. If f is differentiable at $x = a$ then near a the tangent line to f at a is a decent approximation for f.

Definition (Linear approximation)

If the function f is differentiable at a then the **linear approximation** to f at a is

$$L(x) = f(a) + f'(a)(x - a)$$

Example

Let $f(x) = e^x \sin x$

1. Give the linear approximation to f at $a = 0$
   $f'(x) = e^x \sin x + e^x \cos x$ so $f'(0) = e^0 \sin 0 + e^0 \cos 0 = 1$
   $f(0) = e^0 \sin 0 = 0$
   $L(x) = f(0) + f'(0)(x - 0) = 0 + 1(x - 0) = x$
2. Estimate $f(0.04)$
   $f(0.04) \approx L(0.04) = 0.04$
3. Exact value from calculator $e^{0.04} \sin(0.04) \approx 0.04162132987$
4. Compute the percent error using a calculator.

$$\text{percent error} = \frac{100|\text{approx} - \text{exact}|}{\text{exact}} = \frac{100|0.04 - 0.04162132987|}{0.04162132987} \approx 3.895\%$$

Example

Estimate $\sqrt[3]{61}$

1. Let $f(x) = \sqrt[3]{x}$. Choose a number near 61 whose cube root is easy:
2. Give linear approximation to f at $a = 64$
   $f(64) = \sqrt[3]{64} = 4$
   $f'(x) = \frac{1}{3}x^{-2/3}$
   $f'(64) = \frac{1}{3}64^{-2/3} = \frac{1}{3 \cdot 16} = \frac{1}{48}$
   $L(x) = f(64) + f'(64)(x - 64) = 4 + \frac{1}{48}(x - 64)$
   $L(61) = 4 + \frac{1}{48}(61 - 64) = 4 - \frac{1}{16} = \frac{63}{16} = 3.9375$
3. Exact value from calculator $\sqrt[3]{61} \approx 3.9364971831$
4. Compute the percent error using a calculator.

$$\text{percent error} = \frac{100|\text{approx} - \text{exact}|}{\text{exact}} = \frac{100|3.9375 - 3.9364971831|}{3.9364971831} \approx 0.0255\%$$

Differentials

1. Suppose $y = f(x)$ has linear approximation
   $L(x) = f(a) + f'(a)(x - a)$ near a.
2. How should a small change in x affect the value of y?
3. PICTURE
4. $\Delta y = f(a + \Delta x) - f(a) \approx L(a + \Delta x) - L(a) =$
   $f(a) + f'(a)(a + \Delta x - a) - f(a) + f'(a)(a - a) = f'(a)\Delta x$

Definition (differential)

The **differential** $dy = f'(x)dx$ is an estimate of the change in y corresponding to a change in x of size dx.

$$\Delta y = f(x + dx) - f(x) \approx dy = f'(x)dx$$

Example

Estimate the change in volume of a ball if the radius is increased from 1ft to 1.1 ft

1. $V = \frac{4}{3}\pi r^3$
2. $\frac{dV}{dr} = 4\pi r^2$
3. $dV = 4\pi r^2 dr$
4. $dV = 4\pi 1^2(1.1 - 1) = .4\pi \approx 1.25663706144$
5. Exact change in volume is $V(1.1) - V(1) = \frac{4}{3}\pi 1.1^3 - \frac{4}{3}\pi 1^3 \approx 1.38648955778$

Mean Value Theorem

Theorem (Mean Value Theorem)

If f is continuous on $[a, b]$ and differentiable on (a, b) then there is a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

1. MVT says there is at least one point c PICTURE
2. May be many such points c PICTURE
3. Even infinitely many such points c PICTURE
4. Continuity is required PICTURE
5. Differentiability is required PICTURE

Example

Decide if the MVT applies to the given function on the given interval. If so find a point c guaranteed by the the MVT

1. $f(x) = |x|$ on $[2, 4]$: MVT applies. $c = 3$ works (as well as any other number between 2 and 4)
2. $f(x) = |x|$ on $[-2, 3]$: MVT does not apply since f is not differentiable at $x = 0$.
3. $g(x) = e^x + e^{-x}$ on $[0, \ln 2]$: MVT applies
   3.1 $g'(x) = e^x - e^{-x}$
   3.2 $\frac{g(\ln 2) - g(0)}{\ln 2 - 0} = \frac{e^{\ln 2} + \frac{1}{e^{\ln 2}} - (1+1)}{\ln 2} = \frac{2 + \frac{1}{2} - 2}{\ln 2} = \frac{1}{2 \ln 2}$
   3.3

$$\begin{aligned} g'(c) &= \frac{1}{2\ln 2} \\ e^c - e^{-c} &= \frac{1}{2\ln 2} \\ \frac{e^{2c} - 1}{e^c} &= \frac{1}{2\ln 2} \end{aligned}$$

Example

Decide if the MVT applies to the given function on the given interval. If so find a point c guaranteed by the the MVT

3.

$$\begin{aligned} e^{2c} - 1 &= \frac{e^c}{2\ln 2} \\ e^{2c} - \frac{e^c}{2\ln 2} - 1 &= 0 \end{aligned}$$

$e^c = \frac{\frac{1}{2\ln 2} \pm \sqrt{(\frac{1}{2\ln 2})^2 + 4}}{2}$

$c = \ln\left(\frac{\frac{1}{2\ln 2} \pm \sqrt{(\frac{1}{2\ln 2})^2 + 4}}{2}\right) \approx 0.35327$

MVT is very useful for proofs: Recall f is **increasing** if $f(x) < f(y)$ whenever $x < y$ but we've been checking that f is increasing by looking showing $f'(x) > 0$

Theorem

If $f'(x) > 0$ on the interval (a, b) and f is continuous on the interval $[a, b]$ then f is increasing on the interval $[a, b]$.

Proof.

1. Suppose $f'(x) > 0$ on the interval (a, b) and f is continuous on the interval $[a, b]$.
2. Suppose $x, y \in [a, b]$ and $x < y$
3. Then by MVT there is $c \in [x, y]$ such that $f'(c) = \frac{f(y)-f(x)}{y-x}$.
4. $c \in [x, y]$ so $c \in [a, b]$ so $f'(c) > 0$
5. Thus $\frac{f(y)-f(x)}{y-x} = f'(c) > 0$. Also $x < y$ so $y - x > 0$.
6. Thus $f(y) - f(x) > 0$ so $f(y) > f(x)$.

□

Example

An airplane goes from 0 to $70\mathrm{m/s}$ ($156.59\mathrm{miles/h}$) in $20\mathrm{s}$. It's maximum acceleration must be at least what?

1. $v(0) = 0$, $v(20) = 70\mathrm{m/s}$
2. By MVT there is $c \in (0, 20)$ such that $a(c) = \frac{\mathrm{d}v}{\mathrm{d}t}|_{t=c} = \frac{v(20)-v(0)}{20-0} = \frac{70}{20} = 3.5\mathrm{m/s}^2$ (That's $3.5/9.8 \approx .357\mathrm{g}$)

Example

How quickly must a car go from 0 to $27\mathrm{m/s}$ ($60.3973\mathrm{miles/h}$) in order to have a max acceleration of at least $1\mathrm{g} = 9.8\mathrm{m/s}$?

1. $v(0) = 0$, $v(t) = 27\mathrm{m/s}$
2. By MVT there is $c \in (0, t)$ such that $a(c) = \frac{\mathrm{d}v}{\mathrm{d}t}|_{t=c} = \frac{v(t)-v(0)}{t-0} = \frac{27}{t} = 9.8\mathrm{m/s}^2$
3. $\frac{27}{t} = 9.8\mathrm{m/s}^2$ so $t = \frac{27}{9.8} \approx 2.755\mathrm{s}$