Lecture 20 – October 10, 2018

- WH 4 – Due Tues. 10/16
- OH 4.5 & 4.6 – Due tonight
- No office hours today
- Midterm 2 – Thurs. 10/18 6pm-6:55pm in Journalism Building (JR) 300

Today

- l'Hôpital's Rule

L'Hôpital's Rule

Theorem (L'Hôpital's Rule)

Let $a \in \mathbf{R}$ *or* $a = \infty$ *or* $a = -\infty$

1. *If* $\lim_{x\to a} f(x) = 0$ *and* $\lim_{x\to a} g(x) = 0$ *then if* $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ *exists*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

2. *If* $\lim_{x\to a} f(x) = \pm\infty$ *and* $\lim_{x\to a} g(x) = \pm\infty$ *then if* $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ *exists*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Two notes of warning

1. L'Hôpital's Rule is **not the quotient rule** ($\frac{d}{dx}\frac{f(x)}{g(x)} \neq \frac{f'(x)}{g'(x)}$)
2. **must** have $\frac{0}{0}$ form or $\frac{\pm\infty}{\pm\infty}$

Example

Compute the limit

1.

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{2x}{1} = 4$$

2. CHECK:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} x+2 = 4\checkmark$$

Check hypotheses of l'Hôpital's Rule!

Compute the limit

1. $$\lim_{x\to\pi} \frac{\sin x}{x} = \lim_{x\to\pi} \frac{\cos x}{1} = \frac{-1}{1} = -1$$

2. NO!!!!!!!!!!!!!

$$\lim_{x\to\pi} \sin x = 0$$
$$\lim_{x\to\pi} x = \pi$$

So L'Hôpital's Rule doesn't apply.

$$\lim_{x\to\pi} \frac{\sin x}{x} = \frac{0}{\pi} = 0$$

The moral of this problem is: check for $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ indeterminate forms

Example

Compute the limit

1. $$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{x^{-1}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

2. $$\begin{aligned} \lim_{x\to 0} \frac{\sin x^2}{\tan 10x} &= \lim_{x\to 0} \frac{2x\cos x^2}{10\sec^2 10x} \\ &= \lim_{x\to 0} \frac{1}{5} x \cos x^2 \cos^2 10x \\ &= \lim_{x\to 0} \frac{1}{5} \cdot 0 \cdot \cos 0^2 \cos^2 10 \cdot 0 \\ &= 0 \end{aligned}$$

3. $$\lim_{x\to 0^+} \frac{\ln x}{x} = -\infty \qquad \left(\frac{-\infty}{0} \text{ not indeterminant}\right)$$

Definition (Relative growth rates)

f **grows faster than** g (we write $g \ll f$) if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

or equivalently

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M$$

Then we say that f and g have **comparable growth**.

Example

$$\ln x \ll \sqrt{x} \ll x \ll x^2 \ll x^{100} \ll e^x \ll 10^x \ll 100^x \ll x^x$$

Other indetermintate forms

$0 \cdot \infty$, $\infty - \infty$, 1^∞, ∞^0, 0^0.
Each of these forms must be modified in some way to get a $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ form

Example ($0 \cdot \infty$)

1. $\lim_{x\to\infty} x^3 e^{-x} = \lim_{x\to\infty} \frac{x^3}{e^x} = \lim_{x\to\infty} \frac{3x^2}{e^x} =$
   $\lim_{x\to\infty} \frac{6x}{e^x} \lim_{x\to\infty} \frac{6}{e^x} = 0$
2. $\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{x^{-1}}{-\csc x \cot x} =$
   $\lim_{x\to 0^+} \frac{-\sin x \tan x}{x} = \lim_{x\to 0^+} \frac{-\sin x}{x} \lim_{x\to 0^+} \tan x =$
   $\lim_{x\to 0^+} \frac{-\cos x}{1} 0 = 0$

Example ($\infty - \infty$)

Find highest power x^p of x and multiply by $\frac{x^{-p}}{x^{-p}}$

$$\begin{aligned}
\lim_{x\to-\infty} x^2 - \sqrt[3]{x^6+1} &= \lim_{x\to-\infty} \frac{x^{-2}}{x^{-2}} \cdot (x^2 - \sqrt[3]{x^6+1}) \\
&= \lim_{x\to-\infty} \frac{1-(1+x^{-6})^{1/3}}{x^{-2}} \\
&= \lim_{x\to-\infty} \frac{-\frac{1}{3}(1+x^{-6})^{-2/3}(-6)x^{-7}}{(-2)x^{-3}} \\
&= \lim_{x\to-\infty} -(1+x^{-6})^{-2/3}x^{-4} \\
&= \lim_{x\to-\infty} -\frac{1}{(1+x^{-6})^{2/3}x^4} \\
&= \lim_{x\to-\infty} -\frac{1}{(1+x^{-6})^{2/3}(x^6)^{2/3}} \\
&= \lim_{x\to-\infty} -\frac{1}{(x^6+1)^{2/3}} \\
&= 0
\end{aligned}$$

Example (1^∞, ∞^0, 0^0)

Take ln then raise e to power of answer.

1. $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\frac{\ln x}{x}} = e^{\lim_{x\to\infty} \frac{\ln x}{x}} = e^{\lim_{x\to\infty} \frac{x^{-1}}{1}} = e^0 = 1$
2. $\lim_{x\to 0^+} (2x)^x = ?$
3. $\lim_{x\to\infty} \left(\frac{1}{x}\right)^x = ?$

Example (Compound Interest)

Suppose that an interest rate of r per year is paid on principal of P.

1. If the interest is compounded once over a time period of t years what will be the new account balance? $P(1+rt)$
2. If the interest is compounded twice over a time period of t years what will be the new account balance?
   $P(1+\frac{rt}{2})(1+\frac{rt}{2}) = P(1+\frac{rt}{2})^2$
3. If the interest is compounded n times over a time period of t years what will be the new account balance? $P(1+\frac{rt}{n})^n$
4. If the interest is compounded continuously over a time period of t years what will be the new account balance?

$$\begin{aligned}
\lim_{n\to\infty} P(1+\tfrac{rt}{n})^n &= \lim_{n\to\infty} Pe^{n\ln(1+\frac{rt}{n})} \\
&= Pe^{\lim_{n\to\infty} \frac{\ln(1+\frac{rt}{n})}{n^{-1}}} \\
&= Pe^{\lim_{n\to\infty} \frac{(1+\frac{rt}{n})^{-1} rt(-1)n^{-2}}{-n^{-2}}} \\
&= Pe^{\lim_{n\to\infty} \frac{rt}{1+\frac{rt}{n}}} \\
&= Pe^{rt}
\end{aligned}$$

Example

Compute the limit $\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x$

$$\begin{aligned}
\ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x\right) &= \lim_{x\to\infty} \ln\left(1+\frac{1}{x^2}\right)^x \\
&= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x^2}\right) \\
&= \lim_{x\to\infty} \frac{\ln\left(1+x^{-2}\right)}{x^{-1}} \\
&= \lim_{x\to\infty} \frac{\frac{-2x^{-3}}{1+x^{-2}}}{-x^{-2}} \\
&= \lim_{x\to\infty} \frac{2x^{-1}}{1+x^{-2}} \\
&= 0
\end{aligned}$$

$$\ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x\right) = 0$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x = e^0 = 1$$