Lecture 21 – October 15, 2018

- WH 4 – Due tomorrow (10/16)
- Office hours MW 12:30-2pm in Math Tower (MW) 650
- Midterm 2 – Thurs. 6pm-6:55pm in Journalism Building (JR) 300

Today

- Antidifferentiation/Integration

Definition (Antiderivative)

F is an **antiderivative** of f if $F'(x) = f(x)$.

Example

Show that $x^5 + 9x + 2$ is an antiderivative of $5x^4 + 9$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^5 + 9x + 2) = 5x^4 + 9$$

What other functions can have derivative $5x^4 + 9$?

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^5 + 9x + 4\pi) = 5x^4 + 9$$

Theorem

If $F(x)$ is an antiderivative of $f(x)$ in an interval I then every antiderivative of F on the interval I is of the form $F(x) + C$ for some constant C.

Definition (Indefinite Integral)

The **indefinite integral** (or just integral) of $f(x)$ is the set of all antiderivatives of the function f and is denoted

$$\int f(x)\ \mathrm{d}x$$

Example

$$\int \cos x\ \mathrm{d}x = \sin x + C$$

since $\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x$.

Technical note

With definition above

$$\int \frac{1}{x}\ \mathrm{d}x = \begin{cases} \ln|x| + C_1, & x > 0 \\ \ln|x| + C_2, & x < 0 \end{cases}$$

But we will just write $\int \frac{1}{x}\ \mathrm{d}x = \ln|x| + C$.

Every basic derivative or derivative rule gives us a basic antiderivative or antiderivative rule.

Basic Derivative	Basic Integral
$\frac{d}{dx}x^r = rx^{r-1}$	$\int x^p \, dx = \frac{x^{p+1}}{p+1} + C$ if $p \neq -1$
$\frac{d}{dx}e^x = e^x$	$\int e^x \, dx = e^x + C$
$\frac{d}{dx} \ln \lvert x \rvert = \frac{1}{x}$	$\int \frac{1}{x} \, dx = \ln \lvert x \rvert + C$
$\frac{d}{dx} \sin x = \cos x$	$\int \cos x \, dx = \sin x + C$
$\frac{d}{dx} \cos x = -\sin x$	$\int \sin x \, dx = -\cos x + C$
$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} \, dx = \arctan x + C$
$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$
$\frac{d}{dx} \sec^{-1} \lvert x \rvert = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} \lvert x \rvert + C$

Basic Derivative	Basic Integral
$\frac{d}{dx} \tan x = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + C$
$\frac{d}{dx} \cot x = -\csc^2 x$	$\int \csc^2 x \, dx = -\cot x + C$
$\frac{d}{dx} \sec = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + C$
$\frac{d}{dx} \csc = -\csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + C$

Integration Techniques

Derivative Rule	Integration Technique
Const. Mult. Rule $\frac{d}{dx}[cf(x)] = cf'(x)$	Const. Mult. Rule $\int cf(x)\ dx = c\int f(x)\ dx$
Sum rule $\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$	Sum Rule $\int f(x)+g(x)\ dx = \int f(x)\ dx + \int g(x)\ dx$
Product rule $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x)+f(x)g'(x)$	Integration by Parts Section 7.2
Quotient rule $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$	-
Chain rule $\frac{d}{dx}f(g(x)) = f'(g(x))f(x)$	Substitution Section 5.5

Example

Compute the following indefinite integrals THEN CHECK YOUR ANSWER!!

1. $\int \frac{1}{6t}\ dt = \frac{1}{6}\ln|t| + C$
2. $\int \frac{1}{e^{2y}}\ dy = \int e^{-2y}\ dy = \frac{e^{-2y}}{-2} + C$
3. $\int \sqrt[3]{x}\ dx = \int x^{1/3}\ dx = \frac{x^{4/3}}{4/3} + C = \frac{3}{4}x^{4/3} + C$
4. $\int \sec^2 7x\ dx = \frac{1}{7}\tan 7x + C$
5. $\int \frac{1}{49+t^2}\ dt = \frac{1}{49}\int \frac{1}{1+\left(\frac{t}{7}\right)^2}\ dt = \frac{1}{7}\arctan\frac{t}{7} + C$

You are solving a diff eq of the form $y' = f(x)$.

Example

Initial value problems: Acceleration of gravity is $-9.8\,\mathrm{m/s^2}$ A ball is thrown upwards with initial velocity $10\,\mathrm{m/s}$ Give velocity at time t. Suppose initial height of ball is $60\,\mathrm{m}$. Give height at time t.

- $a(t) = -9.8$
- $v(t) = \int a(t)\,\mathrm{d}t = \int 9.8\,\mathrm{d}t = -9.8t + C_1$
- $10 = v(0) = -9.8 \cdot 0 + C_1$ so $C_1 = 10$
- $v(t) = -9.8t + 10$ is the velocity (in $\mathrm{m/s}$) at time t
- $s(t) = \int v(t)\,\mathrm{d}t = \int -9.8t + 10\,\mathrm{d}t = -4.9t^2 + 10t + C_2$
- $60 = s(0) = -4.9 \cdot 0^2 + 10 \cdot 0 + C_2$ so $C_2 = 60$
- $s(t) = -4.9t^2 + 10t + 60$ is the height (in m) at time t

Summation (Sigma) Notation

Sigma notation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

Example (Sigma notation)

Compute the following sums:

1. $\sum_{i=1}^{4} 3 - i^2 = (3-1^2) + (3-2^2) + (3-3^2) + (3-4^2) = (3-1) + (3-4) + (3-9) + (3-16) = 2 - 1 - 6 - 13 = -18$
2. $\sum_{i=4}^{6} \frac{(-1)^i}{i} = \frac{(-1)^4}{4} + \frac{(-1)^5}{5} + \frac{(-1)^6}{6} = \frac{1}{4} - \frac{1}{5} + \frac{1}{6} = \frac{5\cdot 6}{120} - \frac{4\cdot 6}{120} + \frac{4\cdot 5}{120} = \frac{30-24+20}{120} = \frac{26}{120} = \frac{13}{60}$

Example (Sigma notation)

Write the following sums using summation notation:

1. $\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} = \sum_{i=1}^{4} \frac{i}{i+2}$
2. $-1 + 10 - 100 + 1,000 - 10,000 + 100,000 = \sum_{i=0}^{5} 10^i (-1)^{i+1}$