Lecture 23 – October 19, 2018

Today

- Area under curves

Area under curves

1. On Mon. we looked at problems of the form: Find a function $F(x)$ with $F'(x) = \sin 2x + x^4 + 10$.
2. Today we will look at problems of the form: Find the area under the curve $f(x) = \sin 2x + x^4 + 10$ on the interval $[1, 5]$.
3. Next Wed. we will see that these problems are closely related.

1. Recall to get slope of curve at a point (aka derivative)
    1.1 Found slope of secant line on a interval (say $[a, a+h]$)
    1.2 Then took limit of slope of secant line as width, h, of interval went to 0
2. For area between $f(x)$ and x-axis on interval $[a, b]$ we will use same philosophy
    2.1 Approximate area as a bunch of rectangles
    2.2 Take limit as max width of rectangles goes to 0

Sigma notation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

Riemann Sums

Definition (Riemann Sum)

Given a function f defined on the interval $[a, b]$. **Partition** the interval $[a, b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_{n-1}, x_n]$ with $x_0 = a$ and $x_n = b$. Width of kth subinterval is $\Delta x_k = x_k - x_{k-1}$. On each interval $[x_{k-1}, x_k]$ choose a point $x_k^* \in [x_{k-1}, x_k]$. A **Riemann Sum** for f on the interval $[a, b]$ is any sum of the form

$$f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n = \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

In particular,

1. if $x_k^* = x_{k-1} \in [x_{k-1}, x_k]$ then the above sum is a **left Riemann Sum**
2. if $x_k^* = x_k \in [x_{k-1}, x_k]$ then the above sum is a **right Riemann Sum**
3. if $x_k^* = \frac{x_{k-1}+x_k}{2} \in [x_{k-1}, x_k]$ then the above sum is a **midpoint Riemann Sum**

Definition (Regular Partition)

The **regular partition** of the interval $[a, b]$ into n subintervals is the partition $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_{n-1}, x_n]$ with $x_0 = a$ and $x_n = b$ such that width Δx_k of each subinterval is

$$\Delta x = \frac{b-a}{n}$$

Thus for a regular partition we must have

$$\begin{aligned}
x_0 &= a \\
x_1 &= a + \Delta x = a + \tfrac{b-a}{n} \\
&\vdots \\
x_k &= a + k\Delta x = a + \tfrac{k(b-a)}{n} \\
&\vdots \\
x_n &= a + n\Delta x = a + \tfrac{n(b-a)}{n} = b
\end{aligned}$$

Definition (Regular Riemann Sum)

Given a function f defined on the interval $[a, b]$ and $n \in \mathbf{N}$ set $\Delta x = \frac{b-a}{n}$. Also set $x_k = a + k\Delta x = a + \frac{k(b-a)}{n}$ so that $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_{n-1}, x_n]$ is the regular partition of the interval $[a, b]$ into n subintervals. Choose $x_k^* \in [x_{k-1}, x_k]$. A **regular Riemann Sum** for f on the interval $[a, b]$ is any sum of the form

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

1. If $x_k^* = x_{k-1} = a + \frac{(k-1)(b-a)}{n}$ then the above sum is the ***n*th regular left Riemann Sum** $L_n = \sum_{k=1}^{n} f(a + \frac{(k-1)(b-a)}{n}) \left(\frac{b-a}{n}\right)$
2. If $x_k^* = x_k = a + \frac{k(b-a)}{n}$ then the above sum is the ***n*th regular right Riemann Sum** $R_n = \sum_{k=1}^{n} f(a + \frac{k(b-a)}{n}) \left(\frac{b-a}{n}\right)$
3. If $x_k^* = \frac{x_{k-1}+x_k}{2} = a + \frac{(k-\frac{1}{2})(b-a)}{n}$ then the above sum is the ***n*th regular midpoint Riemann Sum** $M_n = \sum_{k=1}^{n} f(a + \frac{(k-\frac{1}{2})(b-a)}{n}) \left(\frac{b-a}{n}\right)$

Summation (Sigma) Notation

Sigma notation

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

Example (Sigma notation)

Compute the following sums:

1. $\sum_{k=1}^{4} 3 - k^2 = (3-1^2) + (3-2^2) + (3-3^2) + (3-4^2) =$
   $(3-1) + (3-4) + (3-9) + (3-16) = 2 - 1 - 6 - 13 = -18$
2. $\sum_{k=4}^{6} \frac{(-1)^i}{i} = \frac{(-1)^4}{4} + \frac{(-1)^5}{5} + \frac{(-1)^6}{6} = \frac{1}{4} - \frac{1}{5} + \frac{1}{6} =$
   $\frac{5\cdot 6}{120} - \frac{4\cdot 6}{120} + \frac{4\cdot 5}{120} = \frac{30-24+20}{120} = \frac{26}{120} = \frac{13}{60}$

Example (Sigma notation)

Write the following sums using summation notation:

1. $\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} = \sum_{i=1}^{4} \frac{k}{k+2}$

Manipulating sums

Summation Rules

1. $\sum_{i=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$
2. $\sum_{i=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

A few useful sums

1. $\sum_{k=1}^{n} c = cn$
2. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
3. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$
5. $\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$ if $r \neq 1$

Example

Evaluate the following sums:

1. $\sum_{i=1}^{1000} 2i = 2\sum_{i=1}^{1000} i = 2 \cdot \frac{1000 \cdot (1000+1)}{2} = 1000 \cdot 1001 = 1,001,000$
2. $\sum_{i=1}^{20}(i^2 - 2i + 1) = \sum_{i=1}^{20} i^2 - 2\sum_{i=1}^{20} i + \sum_{i=1}^{20} 1 =$
   $\frac{20(20+1)(2 \cdot 20+1)}{6} - 2 \cdot \frac{20(20+1)}{2} + 20 \cdot 1 =$
   $\frac{20 \cdot 21 \cdot 41}{6} - 20 \cdot 21 + 20 = 17220/6 - 420 + 20 = 2870 - 400 = 2470$
3. $-1 + 10 - 100 + 1,000 - 10,000 + 100,000 = \sum_{i=0}^{5} 10^i(-1)^{i+1}$
4. $\sum_{i=20}^{30} i^3 = \sum_{i=1}^{30} i^3 - \sum_{i=1}^{19} i^3 = \frac{(30^2)(31^2)}{4} - \frac{(19^2)(20^2)}{4} =$
   $\frac{900 \cdot 961 - 361 \cdot 400}{4} = \frac{864,900 - 144,400}{4} = \frac{720,500}{4} = 180,125$

Example

Give the 4th regular left Riemann Sum L_4 for $f(x) = \sin x$ on the interval $[1, 3]$.

$$L_n = \sum_{k=1}^{n} f\left(a + (k-1) \cdot \tfrac{(b-a)}{n}\right) \tfrac{b-a}{n}$$

$$L_4 = \sum_{k=1}^{4} \sin\left(1 + (k-1)\tfrac{3-1}{4}\right) \tfrac{3-1}{4}$$

$$= \sum_{k=1}^{4} \sin\left(1 + \tfrac{k-1}{2}\right) \tfrac{1}{2}$$

$$= \tfrac{1}{2}\sin 1 + \tfrac{1}{2}\sin\tfrac{3}{2} + \tfrac{1}{2}\sin 2 + \tfrac{1}{2}\sin\tfrac{5}{2}$$

Example

Give the 6th regular midpoint Riemann Sum M_6 for $f(x) = 4x^2 - x$ on the interval $[2, 8]$.

$$M_n = \sum_{k=1}^{n} f\left(a + (k - \tfrac{1}{2}) \cdot \tfrac{(b-a)}{n}\right) \tfrac{b-a}{n}$$

$$M_6 = \sum_{k=1}^{6} \left(4\left(2 + (k - \tfrac{1}{2})\tfrac{8-2}{6}\right)^2 - \left(2 + (k - \tfrac{1}{2})\tfrac{8-2}{6}\right)\right) \tfrac{8-2}{6}$$

$$= \sum_{k=1}^{6} \left(4\left(2 + (k - \tfrac{1}{2})\right)^2 - \left(2 + (k - \tfrac{1}{2})\right)\right)$$

Computing Area with Riemann Sums

Example (Area under a quadratic)

Use the summation rules and summation formula given above to express L_n for $f(x) = x^2$ on the interval $[0, 5]$ in closed form (that is, without a summation). Then compute the exact area by taking $\lim_{n\to\infty} L_n$.

$$L_n = \sum_{i=1}^{n} \left(0 + \tfrac{(k-1)(5-0)}{n}\right)^2 \frac{5-0}{n}$$

$$= \sum_{i=1}^{n} \left(\tfrac{5k-5}{n}\right)^2 \frac{5-0}{n}$$

$$= \sum_{i=1}^{n} \tfrac{125k^2-250k+125}{n^3}$$

$$= \sum_{i=1}^{n} \tfrac{125k^2}{n^3} + \sum_{i=1}^{n} \tfrac{250k}{n^3} + \sum_{i=1}^{n} \tfrac{125}{n^3}$$

$$= \tfrac{125}{n^3} \sum_{i=1}^{n} k^2 + \tfrac{250}{n^3} \sum_{i=1}^{n} k + \tfrac{125}{n^3} \sum_{i=1}^{n} 1$$

Example (*continued*)

$$= \frac{125}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{250}{n^3} \cdot \frac{n(n+1)}{2} + \frac{125}{n^3} \cdot n$$

Thus the exact area under the function $f(x) = x^2$ between $x = 0$ and $x = 5$ is

$$\begin{aligned}
\lim_{n\to\infty} L_n &= \lim_{n\to\infty} \left(\frac{125}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{250}{n^3} \cdot \frac{n(n+1)}{2} + \frac{125}{n^3} \cdot n \right) \\
&= \lim_{n\to\infty} \left(\frac{125(2n^2+3n+1)}{6n^2} + \frac{250(n+1)}{2n^2} + \frac{125}{n^2} \right) \\
&= \lim_{n\to\infty} \left(\frac{125\cdot 2}{6} + \frac{125\cdot 3}{6n} + \frac{125\cdot 1}{6n^2} + \frac{250}{2n} + \frac{250}{2n^2} + \frac{125}{n^2} \right) \\
&= \frac{125\cdot 2}{6} \\
&= \frac{125}{3}
\end{aligned}$$

Thus area under x^2 between $x = 0$ and $x = 5$ is $\frac{125}{3}$.